Midterm II

♡

27. marraskuuta 2018

Theorem 1 (Theorem of Matrix Theory)
Let V, W be finite-dimensional vector spaces of dimension n and m, respectively, and ordered bases $\mathscr{B}, \mathscr{C}$, respectively, and $T : V \to W$ a linear transformation.

Then $\varphi : L(V, W) \to F^{m \times n}$ by $T \to [T]_{\mathscr{B},\mathscr{C}}$ is an isomorphism. In particular, $\dim L(V, W) = mn$.

Todistus. Exercise! □

Note 2
"Watching me get confused in notation doesn't help you learn it". –rse

Theorem 3
Let V, W, X be finite-dimensional vector spaces with ordered bases $\mathscr{B}, \mathscr{C}, \mathscr{D}$, respectively, and linear transformations $T : V \to W$, $S : W \to X$. Then

$$[S \circ T]_{\mathscr{B},\mathscr{D}} = [S]_{\mathscr{C},\mathscr{D}} \cdot [T]_{\mathscr{B},\mathscr{C}}.$$

Todistus. Let $v \in V$. In this case, $[S]_{\mathscr{C},\mathscr{D}}[T]_{\mathscr{B},\mathscr{C}}[v]_{\mathscr{B}} = [S]_{\mathscr{C},\mathscr{D}} \cdot [Tv]_{\mathscr{C}}$, which, by definition, equals $[S(Tv)]_{\mathscr{D}} = [(S \circ T)(v)]_{\mathscr{D}} = [S \circ T]_{\mathscr{B},\mathscr{D}}[v]_{\mathscr{B}}$. □

Corollary 4
Let V, W be finite-dimensional vector spaces over F, $\mathscr{B}, \mathscr{C}$ ordered bases, respectively, and $\dim V = \dim W$. Then

$$(T : V \to W \text{ is an isomorphism}) \iff ([T]_{\mathscr{B},\mathscr{C}} \text{ is invertible}).$$

Moreover, if V is a finite-dimensional vector space over F, $\dim V = n$, and $\mathscr{B}$ an ordered basis for V, then $\varphi : L(V,V) \to \mathbb{M}_n F$ is an isomorphism. Moreover, $\forall T, S \in L(V,V)$, φ satisfies the following:

1. $\varphi(T+S) = \varphi(T) + \varphi(S)$
2. $\varphi(T \circ S) = \varphi(T) \circ \varphi(S)$
3. $\varphi(0_V) = 0_{F^n}$
4. $\varphi(1_V) = I_n$

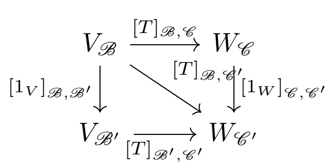
As $L(V,V)$ satisfies all field axioms except (M3) and (M4), it is a (noncommutative) ring, so φ is a **ring isomorphism**.

Theorem 5 (Change-of-basis Theorem)

Let V, W be finite-dimensional vector spaces over F with ordered basis $\mathscr{B}, \mathscr{B}'$ for V, and $\mathscr{C}, \mathscr{C}'$ for W, as well as a linear map $T : V \to W$. Then,

$$\begin{aligned} [T]_{\mathscr{B},\mathscr{C}} &= [1_W]_{\mathscr{C}',\mathscr{C}} [T]_{\mathscr{B}',\mathscr{C}'} [1_V]_{\mathscr{B},\mathscr{B}'} \\ &= [1_W]^{-1}_{\mathscr{C},\mathscr{C}'} [T]_{\mathscr{B}',\mathscr{C}'} [1_V]_{\mathscr{B},\mathscr{B}'} \\ &= [1_W]_{\mathscr{C}',\mathscr{C}} [T]_{\mathscr{B}',\mathscr{C}'} [1_V]^{-1}_{\mathscr{B}',\mathscr{B}}. \end{aligned}$$

Todistus. This is clearly true since the diagram



commutes. □

Note 6

"This is exactly what will come up in real life, whatever real life is..." –rse

Definition 7

Let $A, B \in \mathbb{M}_n(F)$ be matrices. We say that A is **similar** to B and write

$A \sim B$ if $\exists$ an invertible matrix $S \in \mathbb{M}_n(F) \ni A = S^{-1}BS$.

Remark 8. This is an equivalence relation.

Theorem 9

Let $A, B \in \mathbb{M}_n(F)$. Then,

$$(A, B \text{ are similar}) \iff (\exists \text{ an } n\text{-dimensional vector space over } F \text{ with}$$

ordered bases $\mathscr{B}, \mathscr{C} \ni$ linear transformation $T : V \to V$ satisfies $A = [T]_{\mathscr{B}}, B = [T]_{\mathscr{C}}$).

Definition 10

Let $0 \neq V$ be a vector space over F, $T : V \to V$ linear, $\lambda \in F$. Let $S_\lambda : T - \lambda 1_V : V \to V$ linear.

We say that λ is an **eigenvalue** (and write *eit*) of T if S_λ is not one-to-one; i.e., $\ker S_\lambda > 0$, i.e., $\exists 0 \neq v \in \ker S_\lambda$ such that $Tv = \lambda v$.

If we let $E_T(\lambda) := \ker S_\lambda = \{v \in V \mid Tv = \lambda v\}$, then we call it an **eigenspace** of λ for T if it's nonzero.

If $E_T(\lambda) \neq 0$; i.e., an eigenspace; an element $v \in E_T(\lambda)$ is called an **eigenvector** of T relative to λ.

Lemma 11

Let $A, B \in \mathbb{M}_n(F)$ be your favorite matrices, and $A \sim B$ (i.e., $\exists C \in \mathbb{M}_n(F)$ invertible $\ni A = C^{-1}BC$). Then $f_A = f_B$.

Note 12

The converse of this lemma is false in general.

Definition 13

Let V be a finite-dimensional vector space over F, $T : V \to V$ linear, $\mathscr{B}$ an ordered basis for V. Then $f_{[T]_{\mathscr{B}}}$ is called the **characteristic polynomial** of T, written as f_T. By the previous lemma, it's independent of the basis.

Theorem 14

Let V be a finite-dimensional vector space over F, $T : V \to V$ linear. Then the eigenvalues of T are precisely the roots of f_T; i.e., λ is an eit of T iff $f_T(\lambda) = 0$.

Theorem 15 (Cayley-Hamilton)

Let V be a finite-dimensional vector space over F, $T : V \to V$ linear. Then $f_T(T) = 0$; i.e., if $f_T = \sum a_i t^i$, then $\sum a_i T^i = 0$; i.e., $e_T f$=0.

Theorem 16

Let $T : V \to V$ be linear, $\lambda_1, \ldots, \lambda_n \in F$ distinct eits of T. Let $0 \neq v_i \in E_T(\lambda_i)$, $i = 1, 2, \ldots, n$. Then, $\{v_1, \ldots, v_n\}$ is linearly independent.

Corollary 17

Let V be a finite-dimensional vector space over F, $\dim V = n < \infty$. If $T : V \to V$ linear has n distinct eigenvalues, then T is diagonalizeable; i.e., has a basis of eigenvectors.

Corollary 18

Let V be a finite-dimensional vector space over F, $\dim V = n$, $T : V \to V$ linear, then T has at most n distinct eigenvalues.

Corollary 19

Let V be a vector space over F, $\lambda_1, \ldots, \lambda_n$ eigenvalues of $T : V \to V$ linear. Let $v_i \in E_T(\lambda_i)$, $i = 1, \ldots, n$. Then if $v_1 + \ldots + v_n = 0$, then $v_i = 0 \; \forall i$.

Definition 20

Let F be a field, $F \subset \mathbb{C}$, $F = \bar{F}$, V a vector space over F. We say that V is an **inner product space**, and write V is an ips/F, under a map $<,>: V \times V \to F$, written as $\langle v, w \rangle \, for <,> (v, w)$, which satisfies, $\forall v, v_1, v_2, v_3 \in V$, $\forall \alpha \in F$,

1. $\langle v_1 + v_2, v_3 \rangle = \langle v_1, v_3 \rangle + \langle v_2, v_3 \rangle$
2. $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle}$
3. $\langle \alpha v_1, v_2 \rangle = \alpha \langle v_1, v_2 \rangle$
4. $\langle v, v \rangle \in F \cap \mathbb{R}$, and $\langle v, v \rangle \geq 0$ and $= 0$ iff $v = 0$.

Define, $\forall v \in V$, $\|v\| := \sqrt{\langle v, v \rangle} \in \mathbb{R}$. However, this may not lie in F! (for example, consider the length of (1,1) in $\mathbb{Q}^2$ under the dot product).

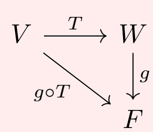
Theorem 21

Let V be a vector space over F. Then $\forall v_1, v_2 \in V$, $\alpha \in F$,

1. $\|v_1\| \geq 0$ and $= 0$ iff $v_1 = 0$.
2. $\|\alpha v_1\| = |\alpha| \|v_1\|$.
3. (Cauchy-Schwarz) $|\langle v_1, v_2 \rangle| \leq \|v_1\| \|v_2\|$.
4. (Minkowski's inequality) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$.

Definition 22

Let V, W be vector spaces over F, $T : V \to W$ linear. Suppose $g \in W^*$, i.e., $g : W \to F$ linear. In other words,



commutes. Observe that $g \circ T \in V^*$. This motivates the following definition:

The **transpose** $T^t : W^* \to V^*$ of T is given by $T^t g := g \circ T$

$T^t : W^* \to V^*$ is linear.

Theorem 23

Let V, W be finite-dimensional vector spaces over F, $\mathscr{B}, \mathscr{C}$ ordered bases for V, W, respectively. Then

$$[T]^t_{\mathscr{B},\mathscr{C}} = [T^t]_{\mathscr{C}^*,\mathscr{B}^*}$$

where $\mathscr{B}^*$ and $\mathscr{C}^*$ are ordered bases dual to $\mathscr{B}$ and $\mathscr{C}$, respectively.

Definition 24

Let V be a vector spaceover F, $\varnothing \neq S \subset V$ a subset of V. The set $S^0 := \{f \in V^* \mid f|_s = 0\} = \{f \in V^* \mid f(s) = 0 \ \forall s \in S\}$ is called the **annihilator** of S.

Lemma 25

Let V be a finite-dimensional vector space over F, and $W \subset V$ be a subspace. Then $\dim V = \dim W + \dim W^0$.

Corollary 26

Let V be a finite-dimensional vector space over F, $W \subset V$ a subspace. Then $W = W^{00}$.

Theorem 27

Let V be an inner product space over F, $S \subset V$ an OR set[a]. Then S is a linearly independent set if $0 \notin S$[b]. If in addition V is a finite dimensional inner product space over F and $V = \text{Span}(S)$, then S is a basis for V and $\dim V = |S|$.

[a] Since it's an OR set, it's nonempty by definition.
[b] Note that this is always true for an ON set.

Theorem 28 (Gram-Schmidt Theorem)

Let V be an inner product space over F, and $S = \{v_1, \ldots, v_n\}$ a linearly independent (nonempty) set in V. Then there exists a set $B = \{y_1, \ldots, y_n\}$ satisfying the following:

1. B is OR
2. $0 \notin B$
3. $\text{Span}(S) = \text{Span}(B)$

4. $y_1 = v_1$

Todistus. We construct B using the *Gram-Schmidt Process.*

Essentially, we induct on n:

Base case ($n = 1$): $v_1 \neq 0$, so $\{v_1\}$ is linearly independent and satisfies 1 through 4.

Inductive case: Assume that if $\{v_1, \ldots, v_n\}$ is linearly independent, then $B_n = \{y_1, \ldots, y_n\}$ OR such that it's linearly independent, $\text{Span}(S_n) = \text{Span}(B_n)$, and $y_1 = v_1$. Let $S_{n+1} = \{v_1, \ldots, v_{n+1}\}$ be linearly independent. We want to show that $\exists B_{n+1}$ satisfying the four requirements.

Define $y_{n+1} := v_{n+1} - \sum_{i=1}^{n} \frac{\langle v_{n+1}, y_i \rangle}{\|y_i\|^2} y_i$. Clearly $\{y_1, \ldots, y_{n+1}\}$ is linearly independent, since y_{n+1} is nonzero. Now we compute $\langle y_{n+1}, y_i \rangle$ for$i = 1, \ldots, n$:

$$\begin{aligned}
\langle y_{n+1}, y_i \rangle &= \left\langle v_{n+1} - \sum_{k=1}^{n} \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} y_k, y_i \right\rangle \\
&= \langle v_{n+1}, y_i \rangle - \sum_{k=1}^{n} \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} \langle y_k, y_i \rangle \\
&= \langle v_{n+1}, y_i \rangle - \sum_{k=1}^{n} \frac{\langle v_{n+1}, y_k \rangle}{\|y_k\|^2} \delta_{ki} \|y_i\|^2 \\
&= \langle v_{n+1}, y_i \rangle - \langle v_{n+1}, y_i \rangle \\
&= 0.
\end{aligned}$$

Hence, $y_{n+1} \perp y_i$, $i = 1, \ldots, n$. Hence, B_{n+1} is an OR set and $0 \notin B_{n+1}$, so B_{n+1} is linearly independent.

Thus, all we have left to show is the span property, which we do as follows:

$$\begin{aligned}
\text{Span}(B_{n+1}) &= \text{Span}(y_1, \ldots, y_{n+1}) \\
&= \text{Span}(v_1, \ldots, v_n, y_{n+1}) \\
&= \text{Span}(v_1, \ldots, v_{n+1}) \qquad \text{(By the \textbf{Replacement Theorem})} \\
&= \text{Span}(S_{n+1}).
\end{aligned}$$

We're done by induction. □

Theorem 29 (ON Theorem)

Let V be a finite-dimensional inner product space over F. Then V has an OR basis $\mathscr{B}$. If $F = \mathbb{R}$ or $\mathbb{C}$, then V has an ON basis $\mathscr{C}$.

Todistus. By the **Gram-Schmidt Theorem**, $\exists \mathscr{B} = \{y_1, \dots, y_n\}$ an OR basis for V.

If $F = \mathbb{R}$ or $\mathbb{C}$, then

$$\mathscr{C} = \left\{ \frac{y_1}{\|y_1\|}, \dots, \frac{y_n}{\|y_n\|} \right\}$$

is an ON basis. □

Note 30

"Anything I wrote down is probably wrong" –rse

Theorem 31 (OR Decomposition Theorem)

Let V be an ips/F, not necessarily finite-dimensional, $S \subset V$ a finite dimensional subspace, $v \in V$. Then,

$$\exists! s \in S, s^\perp \in S^\perp, \ni v = s + s^\perp \tag{*}$$

In particular, $V = S + S^\perp$ and $S \cap S^\perp = 0$; i.e., $V = S \oplus S^\perp$.

Moreover, if $v = s + s^\perp$, $s \in S$, $s^\perp \in S^\perp$, then

$$\|v\| = \|s\|^2 + \|s^\perp\|^2 \tag{Pythagoras}$$

If in addition, V is a finite-dimensional inner product space over F, then $\dim V = \dim S + \dim S^\perp$.

Todistus. Let $\mathscr{B} = \{v_1, \dots, v_n\}$ be an OR basis for S (which exists by the **Gram-Schmidt Theorem**). Let $v \in V$.

Existence: Let $s = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$ be the orthogonal projection of v onto S; $s^\perp := v - s$.

Claim 31.1. $s^\perp \in S^\perp$. Note that this claim implies $V = S + S^\perp$ and $S \cap S^\perp = 0$.

Todistus. Let $j = 1, \dots, n$ be fixed. Then

$$\begin{aligned}
\langle s^\perp, v_j \rangle &= \langle v - s, v_j \rangle && \text{(By definition of } s^\perp) \\
&= \langle v, v_j \rangle - \langle s, v_j \rangle && \text{(By linearity in the first term)} \\
&= \langle v, v_j \rangle - \left\langle \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i, v_j \right\rangle && \text{(By definition of } s) \\
&= \langle v, v_j \rangle - \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} \langle v_i, v_j \rangle && \text{(By linearity in the first term)}
\end{aligned}$$

$$= \langle v, v_j \rangle - \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i \sum_{i=1}^{n} \delta_{ij} \|v_j\|^2$$

$$= 0$$

Hence, if $w = \alpha_1 v_1 + \ldots + \alpha_n v_n$, then $\left\langle s^\perp, w \right\rangle = \sum_{i=1}^n \alpha_i \left\langle s^\perp, v_i \right\rangle = 0$. □

Uniqueness: If $s + s^\perp = r + r^\perp$, $r, s \in S$, $r^\perp, s^\perp \in S^\perp$, then $s - r = r^\perp - s^\perp$, where $s - r \in S$ and $r^\perp - s^\perp = S^\perp$. But $S \cap S^\perp = 0$, so $s - r = r^\perp - s^\perp = 0$, so $r = s = r^\perp = s^\perp = 0$.

Thus, we have showed (*).

Pythagoras: $v = s + s^\perp$, $s \in S$, $s^\perp \in S^\perp$. Then

$$\begin{aligned} \|v\|^2 &= \langle v, v \rangle \\ &= \left\langle s + s^\perp, s + s^\perp \right\rangle \\ &= \langle s, s \rangle + \left\langle s, s^\perp \right\rangle + \left\langle s^\perp, s \right\rangle + \left\langle s^\perp, s^\perp \right\rangle \\ &= \|s\|^2 + 0 + 0 + \|s^\perp\|^2 \\ &= \|s\|^2 + \|s^\perp\|^2. \end{aligned}$$

Finally, if V is a fdips$/F$, then

$$\begin{aligned} \dim S + \dim S^\perp &= \dim \left(S + S^\perp\right) - \dim \left(S \cap S^\perp\right) \end{aligned}$$

(By the **Counting Theorem**)

$$\begin{aligned} &= \dim V - \dim 0 \\ &= V \end{aligned}$$

□

Corollary 32 (Bessel's Inequality)

Let V be an ips$/F$, $B = \{v_1, \ldots, v_n\}$ be an OR set, $0 \notin B$. Let $v \in V$. Then

$$\sum_{i=1}^{n} \frac{|\langle v, v_i \rangle|}{\|v_i\|^2} v_i \leq \|v\|^2$$

with equality holding iff $v = \sum_{i=1}^n \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$.

Note 33

Let V be an ips$/F$, $S \subset V$ a finite-dimensional subspace. If $v \in V$, $\exists! s \in S$, $s^\perp \in S^\perp \ni v = s + s^\perp$. We call s the **orthogonal projection of v on S**, and write v_S for s.

By the proof of the OR decomposition theorem, if $\mathscr{B} = \{v_1, \dots, v_n\}$ is an OR basis for S, then

$$v_S = \sum_{i=1}^{n} \frac{\langle v, v_i \rangle}{\|v_i\|^2} v_i$$

and $s^\perp = v - s = v - v_S$.

Since it's unique, it's *independent of the orthogonal basis*!

Theorem 34 (Approximation Theorem)

Let V be an ips$/F$, $S \subset V$ a finite-dimensional subspace, $v \in V$. Then v_S is closer to V than any other vector in S; i.e., $d(v, v_S) = \|v - v_S\| \leq \|v - r\|$ $\forall r \in S$.

Equivalently, $d(v, v_S) = d(v, S)$.

Moreover, v_S is the unique closest vector in S to v, i.e., if $\|v - v_S\| = \|v - r\|$, $r \in S$, then $r = v_S$.

Todistus. By the **OR Decomposition Theorem**, $v = s + s^\perp$, $s = v_S \in S$, $s^\perp = v - v_S \in S^\perp$.

If $r \in S$, then $v - r = (v - v_S) + (v_S - r) = s^\perp + \underbrace{(v_S - r)}_{\in S}$, so $\langle v - v_S, v_S - r \rangle = 0$.

By Pythagoras, $\|v - r\|^2 = \|v - v_S\|^2 + \|v_S - r\|^2$, which is greater than or equal to $\|v - v_S\|^2$, with equality iff $\|v_S - r\| = 0$; i.e., $v_S = r$. □