Analysis HW #1

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1 Addition

Problem (2.2.1)**.** Prove the following proposition: for any natural numbers a, b, c, we have $(a + b) + c = a + (b + c)$.

Solution. We induct on c.

Base case ($c = 0$):

$$\begin{aligned}(a+b)+0 &= (a+b) && \text{(By \textbf{Lemma 2.2.2})}\\ &= a+b\\ &= a+(b+0) && \text{(By \textbf{Lemma 2.2.2})}\end{aligned}$$

Inductive case: Suppose that $(a + b) + c = a + (b + c)$. We want to show that $(a + b) + c + + = a + (b + c + +)$. Since $(a + b)$ is a natural number, we can apply **Lemma 2.2.3** to get that $(a + b) + c + + = ((a + b) + c) + +$. By the inductive hypothesis, this is equivalent to $(a + (b + c)) + +$. By **Lemma 2.2.3**, this equals $a + (b + c) + +$. Thus, $(a + b) + c + + = a + (b + c + +)$, which closes the induction. So we are done. □

Problem (2.2.2)**.** Prove the following lemma: Let a be a positive number. Then there exists exactly one natural number b such that $b + + = a$.

Solution. Since we're trying to prove a property of the positive naturals, and not all the naturals, we can start our base case at $n = 1$, rather than $n = 0$[1].

Base case ($n = 1$): *Existence*: by definition, $0 + + = 1$, and 0 is a natural number.

[1] Alternatively, we could probably say that our statement $P(n)$ is *if n is a positive natural, then there exists exactly one natural number m such that $m + + = n$*, in which case $P(0)$ would be vacuously true.

Uniqueness: suppose we have two natural numbers a and b, where $a++=1$ and $b++=1$. By **Corollary of Lemma 2.2.3**, we know that $a++=a+1$ and $b++=b+1$; hence, $a+1=1$ and $b+1=1$. In particular, $a+1=b+1$. By **Proposition 2.2.6** (cancellation law), we get that $a=b$. Hence, this natural number is unique.

Inductive step: Suppose that for n, there exists exactly one natural number m such that $m++=n$. We want to show that for $n++$, there exists exactly one natural number p such that $p++=n++$.

Existence: Take $p=n$. Then $p++=n++$ holds.

Uniqueness: Suppose there exist two numbers, a and b, such that $a{+}{+}=n{+}{+}$ and $b++=n++$. Then by transitivity, $a++=b++$. By the **Corollary to Lemma 2.2.3**, we get that $a++=a+1$ and $b++=b+1$, so $a+1=b+1$. By the cancellation law, $a=b$, so $p=a=b$ is a unique choice[2]. □

Problem (2.2.3)**.** Prove **Proposition 2.2.12**; i.e., the following properties about order, given natural numbers a, b, c:

(a) (*Order is reflexive*): $a \geq a$.
(b) (*Order is transitive*): If $a \geq b$ and $b \geq c$, then $a \geq c$.
(c) (*Order is anti-symmetric*): If $a \geq b$ and $b \geq a$, then $a = b$.
(d) (*Addition preserves order*): $a \geq b$ if and only if $a + c \geq b + c$.
(e) $a < b \iff a++ \leq b$.
(f) $a < b \iff b = a + d$ for some positive number d.

Solution. (a). By the definition of addition, $0 + a = a$. Since addition is commutative, $0 + a = a + 0$; hence, $a + 0 = a$. By the definition of an order, $a + 0 \geq a$. But since $a + 0 = a$, by **Lemma 2.2.2** we get that $a \geq a$.

(b). By the definition of order, $a = b + d$ for some natural number d, and $b = c + e$ for some natural number e. Hence, $a + b = (b + d) + (c + e)$. By the associativity of addition, $(b + d) + (c + e) = b + (d + (c + e))$. Hence, $a + b = b + (d + (c + e))$. By the commutativity of addition, $a + b = b + a$; hence, $b + a = b + (d + (c + e))$. Since $(d + (c + e))$ is a natural number, we can use the cancellation law to get $a = (d + (c + e)) = d + (c + e)$. From here we get that

$$\begin{aligned} a &= d + (c + e) \\ &= (d + c) + e && \text{(By associativity)} \\ &= (c + d) + e && \text{(By commutativity)} \end{aligned}$$

[2]We never really used the inductive assumption in this proof—so I guess we never really used the hint in Tao's notes.

$$= c + (d + e). \qquad \text{(By associativity)}$$

Since $d+e$ is a natural number and $a = c+(d+e)$, by the definition of order, we have that $a \geq c$.

(c). By the definition of order, $a = b + c$ for some natural number c and $b = a + d$ for some natural number d. By substituting the expanded value of b, we get that $a = (a + d) + c$, which, by associativity, equals $a + (d + c)$. Hence, $a = a + (d + c)$. By **Lemma 2.2.2**, $a + 0 = a + (d + c)$. By the cancellation law, $0 = d + c$, and by **Corollary 2.2.9**, $d = 0$ and $c = 0$. Hence, $a = b + 0$, so by **Lemma 2.2.2**, $a = b$.

(d). ($\implies$): We prove this using induction on c.

Base case: we want to prove that if $a \geq b$, then $a + 0 \geq b + 0$. By **Lemma 2.2.2**, the second statement is equivalent to $a \geq b$, which follows from the assumption trivially.

Inductive step: Suppose that $a \geq b$, and that $a \geq b \implies a + c \geq b + c$. We want to show that $a + c{+}{+} \geq b + c{+}{+}$. We do this as follows:

$$a + c{+}{+} \geq b + c{+}{+} \iff$$

$$a + c{+}{+} = (b + c{+}{+}) + r \iff \qquad \text{(By definition of order)}$$

$$a + (c + 1) = (b + c{+}{+}) + r \iff \qquad \text{(By \textbf{Corollary to Lemma 2.2.3})}$$

$$a + (c + 1) = b + (c{+}{+} + r) \iff \qquad \text{(By associativity)}$$

$$a + (c + 1) = b + (r + c{+}{+}) \iff \qquad \text{(By commutativity)}$$

$$(a + c) + 1 = (b + r) + c{+}{+} \iff \qquad \text{(By associativity)}$$

$$(a + c) + 1 = (b + r) + (c + 1) \iff \qquad \text{(By \textbf{Corollary to Lemma 2.2.3})}$$

$$(a + c) + 1 = ((b + r) + c) + 1 \iff \qquad \text{(By associativity)}$$

$$(a + c) = ((b + r) + c) \iff \qquad \text{(By cancellation)}$$

$$a + c = b + (r + c) \iff \qquad \text{(By associativity)}$$

$$a + c = b + (c + r) \iff \qquad \text{(By commutativity)}$$

$$a + c = (b + c) + r \iff \qquad \text{(By associativity)}$$

$$a + c \geq b + c \qquad \text{(By definition of order)}$$

Thus, $a + c \geq b + c \implies a + c{+}{+} \geq b + c{+}{+}$, but we supposed that $a \geq b$, and by the inductive hypothesis, $a \geq b \implies a+c \geq b+c$, so $a+c{+}{+} \geq b+c{+}{+}$. This closes the induction, and we are done.

($\impliedby$): This part is easy, and done as follows:

$$a + c \geq b + c \implies$$

$$\begin{aligned}
a + c = (b + c) + r &\implies && \text{(By the definition of order)}\\
a + c = b + (c + r) &\implies && \text{(By associativity)}\\
a + c = b + (r + c) &\implies && \text{(By commutativity)}\\
a + c = (b + r) + c &\implies && \text{(By associativity)}\\
a = b + r &\implies && \text{(By cancellation)}\\
a \geq b & && \text{(By definition of order)}
\end{aligned}$$

(e). ($\implies$): Suppose that $a < b$. Then by definition, $b = a + c$ for some natural number c and $a \neq b$.

Claim 0.1. $c \neq 0$.

Proof. Suppose $c = 0$. Then, we have that $a + 0 = b$. By **Lemma 2.2.2**, we get that $a = b$, a contradiction. ☐

Claim 0.2. $c \geq 1$.

Proof. Essentially, what we want to show is that any positive number can be written in the form $1 + d$. Let's do this by inducting on c, an (arbitrary) positive integer.

Base case ($c = 1$): Set $d = 0$. In this case, $c = 1 = 1 + 0 = 1 + d$ by **Lemma 2.2.2**.

Inductive step: Suppose $c = 1+d$ for some number d. Then $c{+}{+} = (1+d){+}{+}$ by injectivity of the successor function. By commutativity, the right-hand side equals $(d + 1) + +$, which by the definition of addition equals $d + + + 1$. Hence, by commutativity, we get that $c + + = 1 + d + +$, which closes the induction. ☐

Since $c \geq 1$, we can write c as $1 + r$, for some natural number r. By plugging this representation of c into $b = a+c$, we get that $b = a+(1+r)$; by associativity, $b = (a + 1) + r$; by **Corollary to Lemma 2.2.3**, $b = (a + +) + r$; by definition of order, $a + + \leq b$.

($\impliedby$): Suppose that $a{+}{+} \leq b$. By **Corollary to Lemma 2.2.3**, $a+1 \leq b$. By definition of order, $a \leq b$. Thus, by definition of $<$, it suffices to show that $a \neq b$. To do this, suppose that $a = b$. Since we know that $a + 1 \leq b$, we can substitute in the value for a to get $b + 1 \leq b$. By definition of order, this means that $b = (b + 1) + r$ for some natural number r. By associativity, we get $b = b + (1 + r)$; by cancellation, we get that $0 = 1 + r$. By commutativity, $0 = r + 1$. By **Corollary to Lemma 2.2.3**, we get that $0 = r + +$; hence, it is a successor of a natural number. But this is a contradiction of **Axiom 2.3**; hence, our assumption was false, so $a \neq b$, so we are done.

(f). ($\Longrightarrow$): From (e), we know that $a < b \implies a{+}{+} \leq b$. Hence, $b = (a{+}{+}) + r$ by definition of order. By the **Corollary to Lemma 2.2.3**, $b = (a+1) + r$, which by associativity equals $a + (1 + r)$. Set $d = 1 + r$. Clearly, d is positive, since by commutativity and the **Corollary to Lemma 2.2.3**, $d = r{+}{+}$, so by **Axiom 2.3**, $d \neq 0$. Also, $b = a + d$ by substitution, so we are done.

($\Longleftarrow$): Suppose that $b = a + d$ for some positive number d. We proved in (e) that a positive number must be ≥ 1; hence, $d = 1 + r$ by definition of order. By commutativity, $d = r + 1$; by the **Corollary to Lemma 2.2.3**, $d = r{+}{+}$. Substituting this in, we get $b = a + r{+}{+}$, which by commutativity equals $(r{+}{+}) + a$, which by the definition of addition, case II, equals $(r + a){+}{+}$. Since $b = (r + a){+}{+}$ and $(r + a){+}{+} + 0 = (r + a){+}{+}$ by **Lemma 2.2.2**, we get that $b = (r + a){+}{+} + 0$. By the definition of order, this means that $b \geq (r + a){+}{+}$. By (e), this implies that $b > r + a$.

□

Problem (2.2.5)**.** Prove proposition 2.2.14: Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular, this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Solution. Following the hint, we define $Q(n)$ to be the property that $P(m)$ is true for all $m_0 \leq m < n$. We prove the statement by inducting on n.

Base case ($n = 0$): No such $m < 0$ exist, so $Q(0)$ is vacuously true.

Inductive step: suppose $Q(n)$ is true—that is, $P(m)$ is true for all $m_0 \leq m < n$. But since $P(m)$ is true for all $m_0 \leq m < n$, $P(n)$ is true by the assumption in the problem statement. Hence, $P(m)$ is true for all $m_0 \leq m \leq n$; by properties of order, this is equivalent to $P(m)$ being true for all $m_0 \leq m < n + 1$; in other words, $Q(n + 1)$ is true, which closes the induction. □

Problem (2.2.6)**.** Prove the property of backwards induction.

Solution. We show this by induction on n.

Base case ($n = 0$): Suppose that $P(0)$ is true. We want to show that $P(m)$ is true for all $m \leq 0$. The only m that fits this description is $m = 0$[3]. But $P(m)$ is true by assumption.

Inductive step: suppose that if $P(n)$ is true, then $P(m)$ is true for all natural numbers $m \leq n$. It suffices to show that if $P(n{+}{+})$ is true, then $P(m)$ is true for all natural numbers $m \leq n{+}{+}$. But by the given property of the statement P, if $P(n{+}{+})$ is true, then $P(n)$ must be true as well. By the inductive hypothesis, $P(m)$ is true for all natural numbers $m \leq n$. But by assumption it's also true for $n{+}{+}$. Hence, $P(m)$ is true for all natural numbers $m \leq n{+}{+}$, which closes the induction. □

2 Multiplication

Before we begin doing the problems in the multiplication section, it will be helpful to prove a couple of lemmas that we'll use later on[4].

Lemma 1

Let m be a natural number. Then, $m \times 0 = 0$.

Proof. We do this by induction on m.

Base case ($m = 0$): $0 \times 0 = 0$ by the definition of multiplication, case I.

Inductive step: suppose that $m \times 0 = 0$. We want to show that $m{+}{+} \times 0 = 0$. By the definition of multiplication, case II, we know that $m{+}{+} \times 0 = (m \times 0) + 0 = 0 + 0 = 0$, using the inductive hypothesis at the second equality. □

Lemma 2

Let m be a natural number. Then $0 \times m = m \times 0$.

Proof. We prove this by induction.

Base case ($m = 0$): By definition, $0 \times m = 0$. By **Lemma 1**, $m \times 0 = 0$. Hence, $0 \times m = m \times 0$.

Inductive step: suppose that $0 \times m = m \times 0$. We want to show that $0 \times m{+}{+} = m{+}{+} \times 0$. The left-hand side, by the definition of multiplication, case I, equals 0. The right-hand side, by the definition of multiplication, case II,

[3] True by **Corollary 2.2.9** and the definition of order.

[4] Granted, in like one problem, but they're helpful anyways!

equals $(m \times 0) + 0 = 0 + 0 = 0$ by **Lemma 1**. Hence, $0 \times m{+}{+} = m{+}{+} \times 0$, which closes the induction. □

Lemma 3

Let m, n be natural numbers. Then $m \times n{+}{+} = m \times n + m$.

Proof. We prove this by induction on m.

Base case ($m = 0$): we want to show that $0 \times n{+}{+} = 0 \times n + 0$. The left-hand side is 0 by the definition of multiplication, case II. The right-hand side is $0 + 0$, by the definition of multiplication, which also equals 0.

Inductive step: suppose that $m \times n{+}{+} = m \times n + m$. We want to show that $m{+}{+} \times n{+}{+} = m{+}{+} \times n + m{+}{+}$.

First, let's simplify the left-hand side:

$$\begin{aligned}(m{+}{+}) \times (n{+}{+}) &= (m \times n{+}{+}) + (n{+}{+}) && \text{(By def. of multiplication)}\\ &= m \times n + m + n{+}{+} && \text{(By def. of multiplication)}\end{aligned}$$

Now, let's simplify the right-hand side:

$$\begin{aligned}(m{+}{+}) \times n + (m{+}{+}) &= (m \times n) + n + m{+}{+}\\ &= m \times n + n + m{+}{+} && \text{(By def. of multiplication)}\\ &= m \times n + n + m + 1 && \text{(By \textbf{Cor. to Lemma 2.2.3})}\\ &= m \times n + m + n + 1\\ & && \text{(By commutativity of addition)}\\ &= m \times n + m + n{+}{+} && \text{(By \textbf{Cor. to Lemma 2.2.3})}\end{aligned}$$

Haha, now the left-hand side and the right-hand side are equal, which closes the induction. □

Problem (2.3.1)**.** Let m, n be natural numbers. Then $n \times m = m \times n$.

Proof. We prove this by induction on n.

Base case ($n = 0$): $0 \times m = m \times 0$ is true by **Lemma 2**.

Inductive step: suppose that $n \times m = m \times n$. We want to show that $n{+}{+} \times m = m \times n{+}{+}$.

By applying the definition of multiplication, case II, to the left-hand side, we get that $n{+}{+} \times m = (n \times m) + m$. By the inductive hypothesis, this equals $(m \times n) + m$.

Now, let's look at the right-hand side. By **Lemma 3**, $m \times n{+}{+} = m \times n + m$. Haha, the two sides are equal, induction closed. □

Problem. Prove lemma 2.3.3: Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n and m are both positive, then nm is also positive.

Solution. Following Tao's hint, we'll prove the second statement first.

Let n and m be positive natural numbers. We want to show that $n \times m$ is positive. Suffices to show that $n \times m$ is nonzero, by definition of positive. We proved in 2.2.3(e) that if a number is positive, it is greater than or equal to 1; that is, $n = 1 + a$ and $m = 1 + b$ for some natural numbers a, b. Thus, it suffices to show that $(1 + a) \times (1 + b) \neq 0$. First of all, we notice the following:

$$\begin{aligned} n \times m &= (1 + a) \times (1 + b) \\ &= (a + 1) \times (b + 1) && \text{(By commutativity of addition)} \\ &= a{+}{+} \times b{+}{+} && \text{(By the \textbf{Corollary to Lemma 2.2.3})} \\ &= (a \times b{+}{+}) + b{+}{+} && \text{(By the definition of multiplication)} \end{aligned}$$

To show that this does not equal zero, we proceed by contradiction. Suppose that $n \times m$ *does* equal zero; in this case, $(a \times b{+}{+}) + b{+}{+} = 0$ as well. By **Corollary 2.2.9**, this means that $(a \times b{+}{+}) = 0$, and, more importantly, that $b{+}{+} = 0$. But the latter cannot happen by **Axiom 2.3**, so $n \times m$ does not equal zero; hence, if n and m are both positive, then nm is positive.

Now, let's actually prove the lemma.

($\Longrightarrow$): This part is pretty easy using the above result. If $n \times m = 0$, then nm is not positive. If n and m were both positive, then nm would have been positive. But 0 is not positive; hence, n and m are not both positive; in other words, at least one of n, m is equal to zero.

($\Longleftarrow$): We split this into two cases: $n = 0$ and $m = 0$. When $n = 0$, $nm = 0$ by the definition of multplication, case I. When $m = 0$, $nm = 0$ by our fortuitously proved **Lemma 1**[5].

So we're done. □

Lemma 4

For any natural number c, $c \times 1 = c$.

[5] Hey, I guess we *did* use the lemmas more than once!

Proof. Induct on c.

Base case ($c = 0$): $c \times 1 = 0 \times 1 = 0 = c$ by the definition of multiplication, case I.

Inductive step: suppose that $c \times 1 = c$. We want to show that $c{+}{+} \times 1 = c{+}{+}$. We have that

$$\begin{aligned} c++ \times 1 &= (c+1) \times 1 && \text{(By the \textbf{Corollary to Lemma 2.2.3})} \\ &= c \times 1 + 1 \times 1 && \text{(By distributivity)} \\ &= c \times 1 + 1 && \text{(By the definition of multiplication)} \\ &= c + 1 && \text{(By the inductive hypothesis)} \\ &= c++ && \text{(By the \textbf{Corollary to Lemma 2.2.3})} \end{aligned}$$

which closes the induction. □

Problem (2.3.3)**.** Prove proposition 2.3.5: for any natural numbers a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.

Solution. This is easy if we induct on c.

Base case ($c = 0$): $(a \times b) \times 0 = 0$ by **Lemma 1**; $a \times (b \times 0) = a \times 0 = 0$ by **Lemma 1**; hence, $(a \times b) \times c = a \times (b \times c)$ when $c = 0$.

Inductive step: suppose that $(a \times b) \times c = a \times (b \times c)$. We want to show that $a \times (b \times c++) = (a \times b) \times c++$. We do this as follows:

$$\begin{aligned} a \times (b \times c++) &= a \times (c++ \times b) && \text{(By commutativity of multiplication)} \\ &= a \times ((c \times b) + b) && \text{(By the definition of multiplication)} \\ &= a \times (c \times b) + a \times b && \text{(By distributivity)} \\ &= a \times (b \times c) + a \times b && \text{(By commutativity of multiplication)} \\ &= (a \times b) \times c + a \times b && \text{(By the inductive hypothesis)} \\ &= (a \times b) \times c + (a \times b) \times 1 && \text{(By \textbf{Lemma 4})} \\ &= (a \times b) \times (c+1) && \text{(By distributivity)} \\ &= (a \times b) \times c++ && \text{(By the \textbf{Corollary to Lemma 2.2.3})} \end{aligned}$$

which closes the induction; thus, we are done. □

Problem (2.3.4)**.** Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$ for all natural numbers a, b.

Solution. At this point it seems like a grave sin not to use induction, but I claim that we can do it directly[6]. On the left-hand side, we get that

$$\begin{aligned}
(a+b)^2 &= (a+b)^1 \times (a+b) && \text{(By the definition of exponentiation)}\\
&= ((a+b)^0 \times (a+b)) \times (a+b) && \text{(By the definition of exponentiation)}\\
&= (1 \times (a+b)) \times (a+b) && \text{(By the definition of exponentiation)}\\
&= ((a+b) \times 1) \times (a+b) && \text{(By commutativity of multiplication)}\\
&= (a+b) \times (a+b) && \text{(By \textbf{Lemma 4})}\\
&= (a+b) \times a + (a+b) \times b && \text{(By distributivity)}\\
&= a \times a + b \times a + a \times b + b \times b && \text{(By distributivity)}
\end{aligned}$$

On the right-hand side, we get

$$\begin{aligned}
a^2 + 2ab + b^2 &= a^1 \times a + 2ab + b^1 \times b && \text{(By the definition of exponentiation)}\\
&= (a^0 \times a) \times a + 2ab + (b^0 \times b) \times b \\
& && \text{(By the definition of exponentiation)}\\
&= (1 \times a) \times a + 2ab + (1 \times b) \times b \\
& && \text{(By the definition of exponentiation)}\\
&= (a \times 1) \times a + 2ab + (b \times 1) \times b && \text{(By commutativity)}\\
&= a \times a + 2ab + b \times b && \text{(By \textbf{Lemma 4})}\\
&= a \times a + (1+1)(ab) + b \times b \\
&= a \times a + ab + ab + b \times b && \text{(By distributivity)}\\
&= a \times a + b \times a + a \times b + b \times b && \text{(By commutativity)}
\end{aligned}$$

They're equal! So we're done. □

Problem (2.3.5)**.** Prove proposition 2.3.9: let n be a natural number, and let q be a positive number. Then there exist natural numbers m, r such that $0 \le r < q$ and $n = mq + r$.

Solution. We will heed Tao's advice and fix q and induct on n.

Base case ($n = 0$): Take $m = 0$ and $r = 0$. Thus $mq+r = 0\times q+0 = 0+0 = 0$, where we used the definition of multplication, case I. All we need to show is that $r < q$. Since q is positive, we proved in exercise 2.2.3(e) that $q \ge 1$. By definition of order, $1 \ge 0$, so by transitivity of order, $q \ge 0$; hence, $q \ge r$. But $q \ne r$,

[6] In the following chain of equivalences, we're going to be a bit careless with parentheses since we've already proven associativity of multiplication in **Proposition 2.3.5** in the previous exercise—so a couple of times, I might skip an "associativity" step.

for if they were equal, then q would equal zero, and not be positive, which is a contradiction. Hence, $r < q$. But $r = 0$ so $r \geq 0$. Thus, $0 \leq r < q$, as desired.

Inductive step: suppose that $n = mq + r$ where $0 \leq r < q$. We want to show that $n{+}{+} = m'q + r'$ where $0 \leq r' < q$ for some natural numbers m', r'. Suppose that $m' = m$, and $r' = r{+}{+}$. We have two cases: either $r' < q$, or $r{+}{+} \not< q$. In the first case, we get that $m'q + r' = mq + r{+}{+} = r{+}{+} + mq = (r + mq){+}{+} = n{+}{+}$ and $0 \leq r' < q$, so we are done. Thus, we only need to consider the case where $r{+}{+} \not< q$. By trichotomy of order, this means that $r' > q$ or $r' = q$. Assume $r' > q$; that is, $r' = q + a$ and $r' \neq q$. By the inductive hypothesis, $r < q$. By **Proposition 2.2.12(e)**, $r{+}{+} = r' \leq q$. By trichotomy of order, this is a contradiction. Hence, we are left with the case where $r' = q$. But then, $n{+}{+} = m'q + r' = mq + r' = mq + q$, which by distributivity equals $q(m + 1)$, which by the **Corollary to Lemma 2.2.3** and commutativity equals $m{+}{+} \times q$. Hence, we get that $n{+}{+} = m{+}{+} \times q + 0$ by **Lemma 2.2.2**. Hence, we redefine the variables $m' = m{+}{+}$ and $r' = 0$; we can see that $n{+}{+} = m' \times q + r'$ as shown above, and $r' = 0$ so $0 \leq r' < q$ by an argument identical to the one in the base case. So we have shown every case, so we close the induction, and we are done. □

3 Bijections

Problem (3)**.**

Solution. Take $c = 0_{\text{other}}$ and g be the successor function on $\mathbb{N}_{\text{other}}$, denoted $++_{\text{other}}$. By the principle of recursion, there exists a unique function $f : \mathbb{N} \to \mathbb{N}_{\text{other}}$ such that $f(0) = 0_{\text{other}}$ and $(\forall n) f(n{+}{+}) = g(f(n)) = f(n) ++_{\text{other}}$. This f is therefore a bijection since it respects $f(0) = 0_{\text{other}}$ and $(\forall n) f(n{+}{+}) = f(n) ++_{\text{other}}$. □