Analysis HW #2

Dyusha Gritsevskiy

January 2019

Problem (4.1.3)**.** Prove that $(-1) \times a = -a$ for every integer a.

Solution. Let's write the two integers as formal differences, where -1 would be $0 - -1$, and a would be $c - -d$ for some natural numbers c, d. By the definition of multplication on integers, $(-1) \times a = (0 - -1) \times (c - -d) = (0 \times c + 1 \times d) - -(0 \times d + 1 \times c) = (0+d) - -(0+c) == d - -c = -(c - -d) = -a$. □

Problem (4.1.4)**.** Prove the remaining identities in **Proposition 4.1.6**; i.e., that

(a) $x + y = y + x$
(b) $(x + y) + z = x + (y + z)$
(c) $x + 0 = 0 + x = x$
(d) $x + (-x) = (-x) + x = 0$
(e) $xy = yx$
(f) $(xy)z = x(yz)$
(g) $x1 = 1x = x$
(h) $x(y + z) = xy + xz$
(i) $(y + z)x = yx + zx$.

Solution. For all these problems, let $x = a - -b$, $y = c - -d$, and $z = e - -f$.

(a).

$$\begin{aligned} x + y &= (a - -b) + (c - -d) \\ &= (a + c) - -(b + d) && \text{(By definition of addition on } \mathbb{Z}) \\ &= (c + a) - -(d + b) && \text{(By commutativity of addition on } \mathbb{N}) \\ &= (c - -d) + (a - -b) && \text{(By definition of addition on } \mathbb{Z}) \\ &= y + x \end{aligned}$$

(b).

$$\begin{aligned}
(x+y)+z &= ((a--b)+(c--d))+(e--f) \\
&= ((a+c)--(b+d))+(e--f) \\
&\qquad \text{(By definition of addition on } \mathbb{Z}) \\
&= ((a+c)+e)--((b+d)+f) \quad \text{(By definition of addition on } \mathbb{Z}) \\
&= (a+(c+e))--(b+(d+f)) \\
&\qquad \text{(By associativity of addition on } \mathbb{N}) \\
&= (a+b)+((c+e)--(d+f)) \quad \text{(By definition of addition on } \mathbb{Z}) \\
&= (a+b)+((c--d)+(e--f)) \\
&\qquad \text{(By definition of addition on } \mathbb{Z}) \\
&= x+(y+z)
\end{aligned}$$

(c). $x+0=0+x$ by (a); thus, suffices to show that $0+x=x$:

$$\begin{aligned}
0+x &= (0--0)+(c--d) \\
&= (0+c)--(0+d) && \text{(By definition of addition on } \mathbb{Z}) \\
&= c--d && \text{(By definition of addition on } \mathbb{N}) \\
&= x
\end{aligned}$$

(d). $x+(-x)=(-x)+x$ by (a); thus, suffices to show that $x+(-x)=0$:

$$\begin{aligned}
x+(-x) &= (a--b)+(b--a) \\
&= (a+b)--(b+a) && \text{(By definition of addition on } \mathbb{Z}) \\
&= (a+b)--(a+b) && \text{(By commutativity of addition on } \mathbb{N}) \\
&= (a--a)+(b--b) && \text{(By definition of addition on } \mathbb{Z}) \\
&= (0--0)+(0--0) \\
&= 0.
\end{aligned}$$

(e).

$$\begin{aligned}
xy &= (a--b)\times(c--d) \\
&= (ac+bd)--(ad+bc) && \text{(By definition of multiplication on } \mathbb{Z}) \\
&= (ca+db)--(da+cb) && \text{(By commutativity of multiplication on } \mathbb{N}) \\
&= (ca+db)--(cb+da) && \text{(By associativity of addition on } \mathbb{N}) \\
&= (c--d)\times(a--b) && \text{(By definition of multiplication on } \mathbb{Z})
\end{aligned}$$

(f). Proven in Tao

(g). $x1 = 1x$ follows from (e); thus, suffices to show that $1 \times x = x$:

$$\begin{aligned}
1 \times x &= (1 --0) \times (a --b) \\
&= (1 \times a + 0 \times b) --(1 \times b + 0 \times a) \\
&\qquad \text{(By definition of multiplication on } \mathbb{Z}\text{)} \\
&= (a + 0) --(b + 0) \qquad \text{(By \textbf{Lemma 4} on the last homework)} \\
&= a --b \qquad \text{(By properties of addition on } \mathbb{N}\text{)} \\
&= x
\end{aligned}$$

(h).

$$\begin{aligned}
x(y + z) &= (a --b) \times ((c --d) + (e --f)) \\
&= (a --b) \times ((c + e) --(d + f)) \quad \text{(By definition of addition on } \mathbb{Z}\text{)} \\
&= (a \times (c + e) + b \times (d + f)) --(b \times (c + e) + a \times (d + f)) \\
&\qquad \text{(By definition of multiplication on } \mathbb{Z}\text{)} \\
&= (ac + ae + bd + bf) --(bc + be + ad + af) \\
&\qquad \text{(By the distributive property on } \mathbb{N}\text{)} \\
&= (ac + bd + ae + bf) --(bc + ad + be + af) \\
&\qquad \text{(By the commutativity of addition on } \mathbb{N}\text{)} \\
&= ((ac + bd) --(bc + ad)) + ((ae + bf) --(be + af)) \\
&\qquad \text{(By definition of addition on } \mathbb{Z}\text{)} \\
&= (a --b) \times (c --d) + (a --b) \times (e --f) \\
&\qquad \text{(By definition of multiplication on } \mathbb{Z}\text{)}
\end{aligned}$$

(g).

$$\begin{aligned}
(y + z)x &= x(y + z) && \text{(By (e))} \\
&= xy + xz && \text{(By (h))} \\
&= yx + zx && \text{(By (e).)}
\end{aligned}$$

□

Problem (4.1.5)**.** Prove **Proposition 4.1.8**: Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Proof. Write a as a formal difference $n --m$, and b as a formal difference $p --q$. We know that $ab = (n --m) \times (p --q) = 0 --0$; hence, $(np + mq) --(nq + mp) = 0 --0$. Thus, $np + mq = 0$ and $nq + mp = 0$. By **Corollary 2.2.9**, $np = mq = nq = mp = 0$. By **Lemma 2.3.3**, at least one of $\{n, p\}$ is zero, at least one of $\{m, q\}$ is zero, at least one of $\{n, q\}$ is zero, and at least one of $\{m, p\}$ is zero. We can split this into two cases:

Case I ($n = 0$): Since at least one of $\{n, p\}$ is zero, p can either be zero or non-zero.

Case A ($p = 0$): Since at least one of $\{m, p\}$ is zero, m can either be zero or non-zero.

Case ℵ ($m = 0$): In this case, $n - -m = 0 - -0$, so $a = 0$, so we're done.

Case ℶ ($m \neq 0$): Since at least one of $\{m, q\}$ is zero, $q = 0$. Hence, $p - -q = 0 - -0$, so $b = 0$, so we're done.

Case B ($p \neq 0$): Since at least one of $\{m, p\}$ is zero, $m = 0$. Hence, $n - -m = 0 == 0$, so $a = 0$, so we're done.

Case II ($n \neq 0$): Since at least one of $\{n, p\}$ is zero, $p = 0$. Also, at least one of $\{n, q\}$ is zero, so $q = 0$. Hence, $p - -q = 0 - -0$, so $b = 0$, so we're done. □

Problem (4.1.6)**.** Prove **Corollary 4.1.9**: If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Solution. Let's begin by denoting a as $m - -n$, b as $p - -q$, and c as $x - -y$, where $x - -y \neq 0$. By the equivalence of integer representations, it suffices to show that $m + q = n + p$. We do this as follows:

$$ac = bc \implies$$

$$(m - -n) \times (x - -y) = (p - -q) \times (x - -y) \implies$$

$$(mx + ny) - -(my + nx) = (px + qy) - -(qx + py) \implies$$

(By definition of multiplication on $\mathbb{Z}$)

$$mx + ny + qx + py = my + nx + px + qy \implies$$

(By equivalence of integer representations)

$$(m + q)x + (n + p)y = (n + p)x + (m + q)y \implies$$

(By distributivity over $\mathbb{N}$)

$$((m + q)x - -(n + p)x) = ((n + p)y - -(m + q)y) \implies$$

(By equivalence of integer representations)

$$((m + q)x - -(n + p)x) + ((m + q)y - -(n + p)y) = 0 \implies$$

(By subtraction on $\mathbb{Z}$)

$$(m + q)(x + y) - -(n + p)(x + y) = 0 \implies$$

(By definition of addition on $\mathbb{Z}$)

$$(m + q)(x + y) = (n + p)(x + y) \implies$$

$$m + q = n + p$$
(By the cancellation law for natural numbers)

which was to be demonstrated. □

Lemma 1

Additive cancellation law for integers: if a, b, c are integers and $a + c = b + c$, then $a = b$.

Proof. Write a as $m - -n$, b as $p - -q$, and c as $x - -y$. We then have that

$$(m - -n) + (x - -y) = (p - -q) + (x - -y) \implies$$
$$(mx + ny) - -(nx + my) = (px + qy) - -(qx + py) \implies$$
$$mx + ny + qx + py = nx + my + px + qy \implies$$
$$m + q = n + p \implies$$
(By similar argument to previous exercise)
$$m - -n = p - -q \implies$$
$$a = b.$$

□

Lemma 2

"Adding to both sides": if a, b, c are integers and $a = b$, then $a + c = b + c$.

Proof. True by substitution. □

Problem (4.1.7)**.** Prove **Lemma 4.1.11**: Let a, b, c be integers. Then, the following are true:

(a) $a > b$ if and only if $a - b$ is a positive natural number.
(b) (Addition preserves order) If $a > b$, then $a + c > b + c$.
(c) (Positive multiplication preserves order) If $a > b$ and c is positive, then $ac > bc$.
(d) (Negation reverses order) If $a > b$, then $-a < -b$.
(e) (Order is transitive) If $a > b$ and $b > c$, then $a > c$.
(f) (Order trichotomy) Exactly one of the statements $a > b$, $a < b$, or $a = b$ is true.

Solution. (a). ($\Longrightarrow$): Suppose that $a > b$. By definition of order, $a \geq b$ and $a \neq b$. Hence, $a = b + c$ for some natural number c. Suppose that $c = 0$. Then, $a = b + 0_{\mathbb{Z}} = b$ by **Lemma 2.2.2**, a contradiction. Hence, $c \neq 0$, so it is a positive natural number. Since $a = b + c$, and $-b$ is an integer, $a + (-b) = b + c + (-b)$. By commutativity and the definition of subtraction, $a - b = c$. But we showed that c is a positive natural number, so we're done.

($\Longleftarrow$): Suppose that $a - b = c$, where c is a positive natural number (so we can consider it an integer; namely, $c - -0$). By "adding to both sides", we get that $a - b + b = c + b$. By **Exercise 4.1.4(d)**, this means that $a + 0 = c + b$, so $a = c + b$, so by commutativity $a = b + c$. Hence, by definition of order, $a \geq b$. Suppose $a = b$. By **Lemma 2.2.2**, we know that $a + 0 = b + c$, and by substitution $a + 0 = a + c$. By cancellation, $0 = c$, which means that c is not positive, a contradiction. Hence, $a \neq b$, but $a \geq b$, so by definition of order, $a > b$.

(b). Suppose that $a > b$. By (a), this means that $a - b = d$ for some positive natural number d. By **Exercise 4.1.4(d)**, $0 = c + (-c)$. By **Lemma 2.2.2**, $(a-b)+0 = d$. Combining these two statements, we get that $(a-b)+(c+(-c)) = d$. By commutativity we get $(a + c) + (-b + -c) = d$; by the laws of algebra, $(a + c) - (b + c) = d$. Since d is a positive natural number and $a + c$ and $b + c$ are integers, by (a), $a + c > b + c$.

(c). Suppose that $a > b$. By (a), $a - b = d$ for some positive natural d. By left-multiplication on both sides, $c(a - b) = cd$. By distributivity, $ca - cb = cd$. By commutativity, $ac - bc = cd$. By **Lemma 2.3.3**, cd is a positive natural number. Thus by (a), $ac > bc$.

(d). Suppose that $a > b$, so by (a), $a - b = d$ where $d \in \mathbb{N}$, and let's explicitly represent a as $m - -n$, b as $p - -q$, and d as $d - -0$. Hence, we have that

$$
\begin{aligned}
m - -n = p - -q + d - -0 &\implies \\
m - -n = (p + d) - -q &\implies && \text{(By addition on } \mathbb{Z}\text{)} \\
m + q = n + p + d &\implies && \text{(By equivalent representations)} \\
n - -m = q - -(p + d) &\implies && \text{(By equivalent representations)} \\
n - -m = (q - -p) + (0 - -d) &\implies && \text{(By addition on } \mathbb{Z}\text{)} \\
n - -m + d - -0 = (q - -p) + (0 - -d) + (d - -0) &\implies \\
& && \text{(By adding to both sides)} \\
n - -m + d - -0 = q - -p + 0 &\implies && \text{(By laws of algebra)} \\
n - -m + d - -0 = q - -p &\implies && \text{(By laws of algebra)}
\end{aligned}
$$

$$\begin{aligned} -a + d = -b &\implies \\ -b &= -a + d \\ -b &> -a && \text{(By (a).)} \end{aligned}$$

(e). Suppose that $a > b$ and $b > c$. By (a), $a = b + d$ and $b = c + e$, where $d, e \in N$ are positive. By **Lemma 2.?.?**, adding two positive naturals yields a natural, so $d + e$ is positive. Adding the two equations together, we get $a + b = b + d + c + e$, which by commutativity yields $a + b = c + d + e + b$, which by cancellation and associativity yields $a = c + (d + e)$. Since $d + e$ is a positive natural, by (a), $a > c$.

(f). We split this proof into two parts. First, we'll show that no more than one of these statements can be true; then, we'll show that at least one of these statements must be true.

Suppose that $a > b$. In that case, by (a), $a - b = c$ where c is a positive natural.

Suppose that also $a < b$. In that case, by (a), $b - a = d$ where d is a positive natural. Adding the two equations together, we get $a - b + b - a = c + d$, which by the laws of algebra gives us that $0 = c + d$. But $c + d$ is positive since we're adding two positive naturals, so we get a contradiction.

Suppose that also $a = b$. In that case, by substitution we get $a - a = c$; by the laws of algebra, $0 = c$. But c is positive, a contradiction. So we've showed that if $a > b$, then the other two statements cannot be true.

Suppose now that $a < b$. We can identically show that then $a > b$ and $a = b$ cannot be true by repeating the above reasoning with a and b switched. So we've showed that if $a < b$, then the other two statements cannot be true.

Suppose now that $a = b$.

Suppose that also $a < b$. But this is a contradiction by definition.

Suppose that also $a > b$. But this is a contradiction by definition.

Thus, we've shown that if any one of these statements is true, the other ones are necessarily false; hence, no more than one of these three statements can be true.

Now, let's show that at least one of these statements is true.

Consider two integers a and b. Their difference, $a - b$, is another integer c. By **Lemma 4.1.5**, one of the following is true: either $c = 0$, or c is a positive natural n, or c is the negation $-n$ of a positive natural n.

In the first case, $a - b = 0$, so $a - b + b = 0 + b$, so by the laws of algebra $a = b$.

In the second case, $a - b = n$ where n is a positive natural, so by (a), $a > b$.

In the final case, $a - b = (0 - -n)$; as shown earlier, this means that $b - a = n$, so by (a), $b > a$.

Since in every possible case one of the three statements is true, and no more than one of the statements can be simultaneously true, exactly one of the statements $a > b$, $a < b$, or $a = b$ is true. □

Problem (4.1.8)**.** Give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but $P(n)$ is not true for all integers n.

Solution. Consider the property $P(n) : n \geq 0$. Clearly $P(0)$ is true since $0 \geq 0$; similarly, we can show that if $n \geq 0$ then $n++ \geq 0$[1] via the **Corollary to Lemma 2.2.3** and the definition of order. However, not all integers are greater than or equal to 0. □

Problem (4.2.1)**.** Show that the definition of equality for the rational numbers is transitive; that is, if $a//b = c//d$ and $c//d = e//f$, then $a//b = e//f$.

Solution. Suppose that $a//b = c//d$ and $c//d = e//f$. By definition of equality, for the first equivalence, we get that $ad = bc$. By definition of equality for the second equivalence, we get that $cf = de$. Now, consider the statement that $ad = bc$. Multiplying both sides by f, we get that $(ad)f = (bc)f$. By associativity, $(ad)f = b(cf)$, but $cf = de$, so $(ad)f = b(de)$. By associativity and commutativity, $(af)d = (be)d$, so by cancellation we get that $af = be$. By definition of equality, $a//b = e//f$. □

Problem (4.2.3)**.** Prove that the rationals $\mathbb{Q}$ form a field; that is, if x, y, z are rationals, then the following properties hold:

(a) $x + y = y + x$
(b) $(x + y) + z = x + (y + z)$
(c) $x + 0 = 0 + x = x$
(d) $x + (-x) = (-x) + x = 0$
(e) $xy = yx$
(f) $(xy)z = x(yz)$
(g) $x1 = 1x = x$
(h) $x(y + z) = xy + yz$

[1] being vacuously true when $n < 0$

(i) $(y+z)x = yx + zx$

(j) If $x \neq 0$, then $xx^{-1} = x^{-1}x = 1$.

Solution. For the rest of this problem, let $x = a//b$, $y = c//d$, and $z = e//f$ be formal ratio representations of x, y, and z, respectively.

(a).

$$\begin{aligned}
x + y &= (a//b) + (c//d) \\
&= (ad + bc)//(bd) && \text{(By definition of addition over } \mathbb{Q}) \\
&= (da + cb)//(db) && \text{(By commutativity of multiplication over } \mathbb{Z}) \\
&= (cb + da)//(db) && \text{(By associativity of multiplication over } \mathbb{Z}) \\
&= (c//d) + (a//b) && \text{(By definition of addition over } \mathbb{Q}) \\
&= y + x
\end{aligned}$$

(b).

$$\begin{aligned}
(x + y) + z &= ((a//b) + (c//d)) + (e//f) \\
&= (ad + bc)//(bd) + (e//f) && \text{(By definition of addition over } \mathbb{Q}) \\
&= ((ad + bc)f + (bd)(e))//(bdf) \\
& && \text{(By definition of addition over } \mathbb{Q}) \\
&= (adf + bcf + bde)//(bdf) && \text{(By distributivity)} \\
&= (adf + b(cf + de))//(bdf) && \text{(By distribytivity)} \\
&= a//b + ((cf + de)//(df)) && \text{(By definition of addition over } \mathbb{Q}) \\
&= (a//b) + ((c//d) + (e//f)) && \text{(By definition of addition over } \mathbb{Q}) \\
&= x + (y + z).
\end{aligned}$$

(c). By (a), $x + 0 = 0 + x$. Thus, it suffices to show that $0 + x = x$. We do this as follows:

$$\begin{aligned}
0 + x &= (0//1) + (a//b) \\
&= (0b + 1a)//(1b) && \text{(By definition of addition over } \mathbb{Q}) \\
&= a//b && \text{(By rules of algebra over } \mathbb{Z}) \\
&= x.
\end{aligned}$$

(d). By (a), $x + (-x) = (-x) + x$. Thus, suffices to show that $x + (-x) = 0$, which we do as follows:

$$x + (-x) = (a//b) + ((-a)//b) \qquad \text{(By the definition of negation over } \mathbb{Q})$$

$$
\begin{aligned}
&= (ab + b(-a))//(bb) && \text{(By definition of addition over } \mathbb{Q}\text{)} \\
&= (ab - ab)//(bb) && \text{(By commutativity of multiplication over } \mathbb{Z}\text{)} \\
&= 0//(bb) \\
&= 0.
\end{aligned}
$$

(e).

$$
\begin{aligned}
xy &= (a//b) \times (c//d) \\
&= (ac)//(bd) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= (ca)//(db) && \text{(By commutativity of multiplication over } \mathbb{Z}\text{)} \\
&= (c//d) \times (a//b) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= yx.
\end{aligned}
$$

(f).

$$
\begin{aligned}
(xy)z &= ((a//b) \times (c//d)) \times (e//f) \\
&= (ac//bd) \times (e//f) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= (ac)e//(bd)f && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= a(ce)//b(df) && \text{(By associativity of multiplication over } \mathbb{Z}\text{)} \\
&= (a//b) \times (ce//df) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= (a//b) \times ((c//d) \times (e//f)) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= x(yz).
\end{aligned}
$$

(g). By (a), $x1 = 1x$. Thus, suffices to show that $x1 = x$, which we do as follows:

$$
\begin{aligned}
x1 &= (a//b) \times (1//1) \\
&= (a1)//(b1) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= a//b && \text{(By rules of algebra over } \mathbb{Z}\text{)} \\
&= x.
\end{aligned}
$$

(h).

$$
\begin{aligned}
x(y+z) &= (a//b) \times ((c//d) + (e//f)) \\
&= (a//b) \times (cf + de)//(df) && \text{(By definition of addition over } \mathbb{Q}\text{)} \\
&= a(cf + de)//b(df) && \text{(By definition of multiplication over } \mathbb{Q}\text{)} \\
&= (acf + ade//bdf) \times 1 && = (acf + ade//bdf) \times (b//b) \\
& && \text{(By (g))}
\end{aligned}
$$

$$= (b(acf + ade))//b(bdf) \qquad \text{(By definition of multiplication over } \mathbb{Q}\text{)}$$
$$= bacf + bade//bbdf \qquad \text{(By distributivity over } \mathbb{Z}\text{)}$$
$$= ((ac)(bf) + (bd)(ae))//(bd)(df)$$
$$\text{(By commutativity of multiplication over } \mathbb{Z}\text{)}$$
$$= ac//bd + ae//bf \qquad \text{(By definition of addition over } \mathbb{Q}\text{)}$$
$$= (a//b) \times (c//d) + (a//b) \times (e//f)$$
$$\text{(By definition of multiplication over } \mathbb{Q}\text{)}$$
$$= xy + xz.$$

(i).

$$\begin{aligned}(y+z)x &= x(y+z) && \text{(By (e))}\\ &= xy + xz && \text{(By (h))}\\ &= yx + zx. && \text{(By (e))}\end{aligned}$$

(j). By (e), $xx^{-1} = x^{-1}x$. Hence, suffices to show that $xx^{-1} = 1$, which we do as follows:

$$\begin{aligned}xx^{-1} &= (a//b) \times (b//a) && \text{(By definition of the reciprocal in } \mathbb{Q}\text{)}\\ &= ab//ba && \text{(By definition of multiplication over } \mathbb{Q}\text{)}\\ &= ab//ab && \text{(By commutativity of multiplication over } \mathbb{Z}\text{)}\\ &= 1 && \text{(By equivalent representations of rationals.)}\end{aligned}$$

□

Lemma 3

The only integer whose negation is itself is 0.

Solution. It is easy to see that $-0 = 0 - -0 = 0 - -0 = 0$, so the negation of 0 is 0.

Suppose that there is some other integer $a - -b$ whose negation is itself; i.e., $a - -b = b - -a$. By algebraic properties of integers, $a - -b + b - -a = 0$, so $(a + b) - -(b + a) = 0$. By commutativity of naturals, $(a + b) - -(a + b) = 0$. Thus, $a + b = 0$. Thus, $a = 0$ and $b = 0$. Thus, $a - -b = 0$. Thus, the only integer whose negation is itself is 0. □

Problem (4.2.4)**.** Prove **Lemma 4.2.7**: Let x be a rational number. Then exactly one of the following three statements is true: (a) x is equal to 0. (b) x is a positive rational number. (c) x is a negative rational number.

Solution. First, let's show that at most one of the three statements can be true.

Suppose that x is a positive rational number. Then, $x = a//b$ for some positive integers a, b.

Suppose that in addition, x is a negative rational number; that is, $x = (-c)//d$ for some positive integers c, d. Hence, $a//b = (-c)//d$. By the equivalence of rationals, $ad = -(c)b = -(cb)$. But a, b, c, d are all positive integers. Thus ad and cb are positive integers. But $-(cb)$ is the negation of a positive integer; hence, negative. Thus, $ad = -(cb)$ implies that a positive integer equals a negative integer, a contradiction.

Suppose that in addition, x is equal to zero. Then $x = 0//1$. By equivalence of rationals, $a//b = x = 0//1 \implies a1 = b0$. Thus, $a = 0$. But a was a positive integer, so this is a contradiction.

Thus, when x is a positive rational, the other two statements cannot be true.

Now suppose that x is a negative rational number. Then, $x = (-c)//d$ for some positive integers c, d. Identical arguments as above hold to show that the other two statements cannot be true.

Finally, suppose that $x = 0$. Then $x = 0//1$.

Suppose that in addition, x is a positive rational number; that is, $x = a//b$ for some positive integers a, b. By equivalence of rationals, $a//b = x = 0//1 \implies a1 = b0$. Thus, $a = 0$. But a was a positive integer, so this is a contradiction.

Suppose that in addition, x is a negative rational number; that is, $x = (-c)//d$ for some positive integers c, d. By equivalence or rationals, $-c//d = x = 0//1 \implies -c1 = d0$. Thus, $-c = 0$, so $c = 0$. But c was a negative integer; in particular, it was non-zero, a contradiction.

Thus, we have shown that no more than one of the three statements can be true at any time.

Now, let's show that at least one of the statements is true at any given time. Suppose that $x = a//b$ for some *arbitrary* integers a, b[2]. By trichotomy of integers, we have the following cases:

Case I ($a > 0$): Thus a is a positive integer. We have several cases for b:

Case A ($b > 0$): In this case, a, b are positive integers, so x is a positive rational number.

Case B ($b < 0$): In this case, a is a positive integer, and b is a negative integer. In particular, b is the negation $-c$ of some positive integer c. Hence,

[2]Of course, $b \neq 0$.

$x = a//(-c)$. By equivalence of rationals, $x = (-a)//c$[3]. Since a and c are positive integers, x is a negative rational number.

Case II ($a < 0$): This a is a negative integer; in particular, it is the negation $-d$ of a positive integer d. What b can be falls into two cases.

Case A ($b > 0$): In this case, $x = (-d)//b$ where b, d are positive integers, so x is a negative rational number.

Case B ($b < 0$): In this case, $x = (-d)//(-b)$. By equivalence of rationals, $x = d//b$. Since d and b are positive integers, x is a positive rational number.

Case III ($a = 0$): In this case, $x = 0//b$. By equivalence or rationals, $x = 0//1$, so x is equal to zero.

Thus, in any case, at least one of the statements is true.

Since it is always true that at least one of the statements is true and at most one of the statements is true, exactly one of the statements is true at all times. □

Problem (4.2.6)**.** Show that if x, y, z are rational numbers such that $x < y$ and z is *negative*, then $xz > yz$.

Solution. Denote x as $a//b$, y as $c//d$, and z as $(-e)//f$, where e, f are positive integers. We know that $y - x = m$, where m is a positive rational number, since $y > x$ and that's how the definition of order works. Then we get that

$$\begin{aligned}
y - x = m &\implies \\
(y - x)(e//f) = m(e//f) &\implies \\
y(e//f) - x(e//f) = m(e//f) &\implies \\
(c//d)(e//f) - (a//b)(e//f) = m(e//f) &\implies \\
ce//df - ae//bf = m(e//f) &\implies \\
(-c)(-e)//df - (-a)(-e)//bf = m(e//f) &\implies \\
(-c//d)(-e//f) - (-a//b)(-e//f) = m(e//f) &\implies \\
(-y)(z) - (-x)(z) = m(e//f) &\implies \\
xz - yz = m(e//f). &
\end{aligned}$$

But since $e//f$ is a positive rational since e is a positive integer and f is a positive integer, then by definition of ordering of the rationals, $xz > yz$. □

[3]To verify this, observe that $ac = (-a)(-c)$.