Analysis HW #5

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Problem (6)**.** Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?

Solution. First, let's prove that E' is closed. To do that, consider some point x that is a limit point of E'. We need to show that $x \in E'$. Since x is a limit point of E', $\forall \epsilon > 0$ such that $N_\epsilon(x)$ intersects non-trivially with E'. Consider a point in this intersection, call it y. If $y = x$, then $x \in E'$, so we're done. Thus assume $y \neq x$. Since y is a limit point of E, $\forall \delta > 0$, $N_\delta(y)$ intersects non-trivially with E. Consider a point in this intersection, call it z. For x to be a limit point of E, $\forall r > 0$, there must be a point $z \in E$ such that $d(x, z) < r$. But if we choose $\epsilon = \delta = r/3$, then by the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < \delta + \epsilon = 2r/3 < r$, so x is a limit point of E, so $x \in E'$.

Now, let's prove that E and $\bar{E}$ have the same limit points. By definition, any limit point of E is also a limit point of $\bar{E}$. Thus, it suffices to show that all limit points of $\bar{E}$ are limit points of E. Consider some limit point x of $\bar{E}$. Then for any $\epsilon > 0$, there is some $y \in \bar{E}$ such that $d(x, y) < \epsilon$. By definition of union, either $y \in E$ or $y \in E'$. In the former case, x is a limit point of E, so we are done. In the latter case, for any $\delta > 0$, there is some $z \in E$ such that $d(y, z) < \delta$. We want to show that x is a limit point of E; namely, that for all $r > 0$, we can find a point z in E such that $d(x, z) < r$. Let $\epsilon = \delta = r/3$. Then by the triangle inequality, $d(x, z) \leq d(x, y) + d(y, z) < \delta + \epsilon = 2r/3 < r$, so x is a limit point of E.

Suppose E and E' have the same limit points. The limit points of E are just E'. So for E and E' to have the same limit points, the limit points of E' must be E'. We know that E' is closed, so all its limit points are in E'; however, we also need to show that every $x \in E'$ is a limit point for E'. This is definitely not true in general–consider the set $\{\frac{1}{n} \mid n \in \mathbb{N}\}$[1]. Its limit point is $\{0\}$, but its set of limit points clearly has no limit points. $\square$

Problem (7)**.** Let $A_1, A_2, A_3, \ldots$ be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B_n} = \cup_{i=1}^n \overline{A_i}$, for $n = 1, 2, 3, \ldots$...
(b) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supseteq \cup_{i=1}^\infty \overline{A_i}$.

Show, by an example, that this inclusion can be proper.

Lemma 1

$B_n' = \cup_{i=1}^n A_i'$.

Proof. Essentially, what we are trying to show here is that $(\cup_{i=1}^n A_i)' = \cup_{i=1}^n A_i'$.

First, let's show that $(\cup_{i=1}^n A_i)' \supseteq \cup_{i=1}^n A_i'$. Suffices to show that for each k from 1 to n, $A_k' \subset (\cup_{i=1}^n A_i)'$, since if each of them is a subset, then their union would also be a subset. But clearly this is true, since the limit points of a set are the limit points of any larger set containing the set. Thus $(\cup_{i=1}^n A_i)' \supseteq \cup_{i=1}^n A_i'$.

Now, all we need to show is that $(\cup_{i=1}^n A_i)' \subseteq \cup_{i=1}^n A_i'$. We can do this via induction; clearly it's true for the $n = 1$ case. Now suppose that $(\cup_{i=1}^n A_i)' \subseteq \cup_{i=1}^n A_i'$; we want to show that $(\cup_{i=1}^n A_i \cup A_{n++})' \subseteq \cup_{i=1}^n A_i' \cup A_{n++}'$. We can show the contrapositive by taking some $x \notin \cup_{i=1}^n A_i' \cup A_{n++}'$. Then there is a neighborhood $N_r(x)$ such that $(\cup_{i=1}^n A_i') \cap N_r(x) \subseteq \{x\}$, and a neighborhood $N_s(x)$ such that $A_{n++}' \cap N_s(x) \subseteq \{x\}$. Since neighborhoods are open sets and the union of open sets is an open set, $N_r(x) \cap N_s(x)$ gives a neighborhood $N_t(x)$ around x. Then definitely $N_t(x) \cap (\cup_{i=1}^n A_i \cup A_{n++}) = \{x\}$, so x is not a limit point for $(\cup_{i=1}^n A_i \cup A_{n++})$ and thus is not in $(\cup_{i=1}^n A_i \cup A_{n++})'$. This closes the induction, so we are done. $\square$

Solution. (a). $\overline{B_n} = B_n \cup B_n' = \cup_{i=1}^n A_i \cup B_n' = \cup_{i=1}^n A_i \cup \cup_{i=1}^n A_i'$, which by rearrangement equals $\cup_{i=1}^n (A_i \cup A_i') = \cup_{i=1}^n \overline{A_i}$.

(b). By definition, $B \supseteq A_i$ for all $i \in \mathbb{N}^+$; using part (a), $\overline{B} \supseteq \overline{A_i}$ for all $i \in \mathbb{N}^+$; combining the two, we get that $\overline{B} \supseteq \cup_{i=1}^\infty \overline{A_i}$.

For the last part of the question, consider $A_i = \{q_i\}$, where q_i is the i^{th} rational number enumerated via a bijection $f : \mathbb{N} \to \mathbb{Q}$. Then $B = \mathbb{Q}$, so $\overline{B} = \mathbb{R}$, whereas $\cup_{i=1}^n \overline{A_i} = \cup_{i=1}^n \overline{q_i} = \cup_{i=1}^n q_i = \mathbb{Q}$. So clearly $B \supsetneq A$. $\square$

[1] Man, we use this set in every example.

Problem (8)**.** Is every point of every open set $E \subseteq \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets of $\mathbb{R}^2$.

Solution. Consider some point $x = (x_1, y_1)$ in an open set E. By definition x is an interior point; that is, $\exists r > 0$ such that $N_r(x) \subseteq E$. Consider the point $y = (x_1 + s, y_1)$ for some $0 < s < r$. By the Euclidean metric, $d(x, y) < r$, so $y \in N_r(x)$, so $y \in E$. Thus, for every $t > 0$, if we choose s to be any positive real such that $0 < s < \min(r, t)$, then y is in the neighborhood of E, and since t is nonzero, $y \neq x$. Hence x is a limit point of E.

This is not true, in general, for a closed set E. Consider the set $E = \{0\}$, which is closed in $\mathbb{R}^2$. However, 0 is not a limit point of E, since in fact E has no limit points. □

Problem (10)**.** Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (p \neq q) \\ 0 & (p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution. (1). (d is symmetric) If $p = q$, then $d(p,q) = \begin{cases} 1 & (p \neq q) \\ 0 & (p = q) \end{cases}$. If $q = p$, then $d(q,p) = \begin{cases} 1 & (q \neq p) \\ 0 & (q = p) \end{cases} = \begin{cases} 1 & (p \neq q) \\ 0 & (p = q) \end{cases} = d(q,p)$.

(2). (d is positive semidefinite) Since d can take on values of either 0 or 1, $d(p,q) \geq 0$ since $1 \geq 0$ and $0 \geq 0$.

Claim 1.1. $d(p,q) = 0 \iff p = q$.

Proof. ($\implies$): Suppose $d(p,q) = 0$. Suppose for contradiction that $p \neq q$. Then $d(p,q) = 1 \neq 0$, a contradiction. got 'em!

($\impliedby$): Suppose that $p = q$. Then $d(p,q) = 0$. □

(3). (triangle inequality) We want to show that $d(x,z) \leq d(x,y) + d(y,z)$. Suppose that $x = z$. Then $d(x,z) = 0$. By (2), $d(x+y) \geq 0$ and $d(y,z) \geq 0$. Hence $d(x,y) + d(y,z) \geq 0 = d(x,z)$, as desired.

Suppose now that $x \neq z$. Then $d(x,z) = 1$. Consider y. If $x = y$, then $y \neq z$. Then $d(x,y) + d(y,z) = 0 + 1 = 1 \geq 1 = d(x,z)$. If $y = z$, then $y \neq x$. Then $d(x,y) + d(y,z) = 1 + 0 = 1 \geq 1 = d(x,z)$. If $x \neq y$ and $z \neq y$, then $d(x,y) + d(y,z) = 1 + 1 = 2 \geq 1 = d(x,z)$.

Consider any singleton set in this metric space, $\{x\}$. Consider the neighborhood $N_{\frac{1}{2}}(x) = \{x\} \subseteq \{x\}$. Since any set is a union of one-point sets, any subset of the metric space is an open set. Every subset is the complement of a union of one-point sets; hence, every subset is closed. Hence, all subsets are both open and closed. All finite sets are definitely compact. Infinite sets are not necessarily compact—consider the covering defined by the union of open covers covering one point each. There's no way to reduce this to a finite subcover. □

Problem (12)**.** Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \ldots$. Prove that K is compact without the Heine-Borel theorem.

Solution. First, K is bounded above by 2. To see this, suppose there's some $x \in K$ where $x > 2$. Then $x = 1/n$ for some natural n, so $1/n > 2$, so $1 > 2n$ so $n < 1/2$, so n must equal 0, a contradiction. Second, K is bounded below by 0 via a similar argument. Hence, $K \subseteq [0, 2]$. We proved in class that any closed interval in $\mathbb{R}$ is compact, so $[0, 2]$ is compact. We showed in **Problem 5** that K has exactly one limit point, which is 0; furthermore, $0 \in K$, so K is closed. By **Theorem 2.35** in Rudin, every closed subset of a compact set is compact. Hence, K is compact. □

Problem (14)**.** Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution. Take the open cover $\cup_{n=1}^{\infty}(\frac{1}{n}, 1)$. It is definitely a cover, since for any $k \in (0, 1)$, $k = 1/r$ for some $r > 1$. Take any $n > r$, so $1/n < 1/r < k$, so $(1/n, 1) \ni k$. However, no finite subset of the open cover covers all of $(0, 1)$, since if it is finite, there is a maximal n, so consider any $m \in (0, 1)$ such that $m < \frac{1}{n}$, which is not covered by the subcover. □

Problem (22)**.** A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.

Solution. Consider the set $K = \{(a_1, a_2, \ldots, a_k) \mid a_i \in \mathbb{Q}\ \forall i\}$. Clearly K is countable, since the rationals are countable, and K is a subset of $\mathbb{R}^k$. Thus, suffices to show that K is dense in $\mathbb{R}^k$; i.e., every point in $\mathbb{R}^k$ is either a limit point of K, or a point in K. To do this, it suffices to show that for a point $x \in \mathbb{R}^k$, any neighborhood of x has a point in K in it. Let $x = (s_1, s_2, s_3, \ldots, s_k)$ where $s_i \in \mathbb{R}\ \forall i$. We want to show that $\forall r > 0$, exists $s \in K$ such that $d(s, x) < r$. Take some $m < \frac{r}{\sqrt{k}}$. By density or the rationals in the reals, choose some rational k_i's such that $s_i < k_i < s_i + m$. Let $s = (k_1, k_2, \ldots, k_k)$. Then $d(s, x) < \sqrt{(s_1 - s_1 - m)^2 + (s_2 - s_2 - m)^2 + \ldots + (s_k - s_k - m)^2} = \sqrt{km^2} =$

$m\sqrt{k} < \frac{r}{\sqrt{k}}\sqrt{k} = r$, so there exists some $s \in K$ such that $d(s, x) < r$, as desired. □

Problem (24)**.** Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ. Take $\delta = 1/n$, $n \in \mathbb{N}^+$, and consider the centers of the corresponding neighborhoods.

Proof. Let's follow the hint and show that this process stops. Suppose this process never stops; in other words, we can always find a point in X that is greater than δ away from any set of points in X. Then consider the neighborhood $N_{\delta/2}(x)$ around any point x in the set. By construction, it contains no points in the set other than x; hence it is not a limit point. Since the choice of x is arbitrary, the set has no limit points, but since it is infinite, that is a contradiction. Hence, the process stops at some point, and we have constructed a finite cover of X via finitely many neighborhoods of radius δ. Consider the centers of the corresponding neighborhoods for $\delta = 1/n$ where $n \in \mathbb{N}^+$ (forming a cover of X). Any $x \in X$ is in the neighborhood of one of those centers for some large $n > \frac{1}{\delta}$, so the it is within a distance δ of the center, so it is a limit point of the centers, and hence the set of centers of the neighborhoods is dense in X. □