Analysis HW #7

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Problem (1)**.** Prove that a metric space X is connected iff it cannot be written as the union of two disjoint non-empty open sets.

Solution. ($\Longrightarrow$): Suppose a metric space X is connected. We want to show that X cannot be written as the union of two disjoint non-empty open sets. Suppose for contradiction that it can be written as the union of two disjoint non-empty open sets; i.e., $X = A \cup B$, where $A \neq \varnothing$, $B \neq \varnothing$, and $A \cap B = \varnothing$. Since A and B are disjoint and $X = A \cup B$, $B = A^{\mathsf{c}}$. Since A is open, B is therefore both closed and open in X. Similarly, since A is the complement of B, A is also both open and closed in X. Since A is closed, $\overline{A} = A$; similarly, $\overline{B} = B$. Since $A \cap B = \varnothing$, we get that $A \cap \overline{B} = \varnothing$, and $\overline{A} \cap B = \varnothing$. Hence A and B are separated sets, but $X = A \cup B$, hence X is not connected, a contradiction. got 'em!

($\Longleftarrow$): Suppose a metric space X cannot be written as the union of two disjoint non-empty open sets. We want to show that X is a connected metric space. We will show the contrapositive: if X is not a connected metric space, then it *can* be written as the union of two disjoint non-empty open sets. Thus, suppose X is not connected. We want to show that X can be written as the union of two disjoint non-empty open sets. By the definition of connectedness, X can be written as a union of two nonempty separated sets. Thus, let $X = A \cup B$ where A, B are nonempty, and $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. Clearly A and B are disjoint, since $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$; by definition they are both nonempty. Thus suffices to show that A and B are open. Suppose A is not open. Since A, B are disjoint and $X = A \cup B$, $B = A^{\mathsf{c}}$ in X. Hence B is not closed. Hence its closure is nonempty. Since A is the complement of B, we can find some $z \in B'$ such that $z \in A$. But $z \in B'$, so $z \in \overline{B}$. Hence $z \in A \cap \overline{B}$, a contradiction. got 'em! $\square$

Problem (2)**.** x is a boundary point of A if it belongs to $\overline{A} \cap \overline{X - A}$. Suppose X is connected. Prove that every $\varnothing \neq A \subsetneq X$ has a boundary point.

Solution. Consider some nonempty $A \subsetneq X$. Then we know that $X - A$ is the complement of A, and since $A \neq X$, $X - A = A^{\mathsf{c}}$ is nonempty.

Claim 0.1. *A and A^{c} are not separated.*

Proof. Suppose for contradiction that A and A^{c} are separated. Then $X = A \cup A^{\mathsf{c}}$, so X is a union of two nonempty separated sets. Thus X is not connected, a contradiction. got 'em! $\square$

Since A and A^{c} are not separated, at least one of the following two statements is false:

1. $A \cap \overline{A^{\mathsf{c}}} = \varnothing$
2. $\overline{A} \cap A^{\mathsf{c}} = \varnothing$

This splits the problem up nicely into two cases.

Case I (① is false): In this case, $A \cap \overline{A^{\mathsf{c}}}$ is nonempty; that is, there is something in A that is also in the closure of the complement of A. Hence, $A \cap \overline{X - A}$ is nonempty. Since $A \subseteq \overline{A}$, $\overline{A} \cap \overline{X - A}$ is nonempty as well. Thus A has a boundary point.

Case II (② is false): In this case, $\overline{A} \cap A^{\mathsf{c}}$ is nonempty, that is, there is something in $X - A$ that is also in the closure of A. Hence $\overline{A} \cap X - A$ is nonempty. But $X - A \subseteq \overline{X - A}$, so $\overline{A} \cap \overline{X - A}$ is nonempty as well. Thus A, once again, has a boundary point, and as are done. $\square$

Problem (3)**.** Prove carefully that $\mathbb{R}^2$ is not a (countable) union of sets S_i, $i = 1, 2, \dots$ with each S_i being a subset of some straight line L_i in $\mathbb{R}^2$.

Solution. Consider the unit circle S^1 in $\mathbb{R}^2$. We know that S^1 contains uncountably many points. (To see this, consider the two-dimensional stereographic projection from S^1 onto the line $y = -1$. This bijectively maps every point from the circle, except for one, onto $\mathbb{R}^1$. Hence the cardinality of S^1 is the same as that of $\mathbb{R}^1$; hence, the number of points in the unit circle is uncountable.) Now, consider some subset S_k for $k \in \mathbb{N}$. Since S_k is a straight line, it intersects S^1 in zero, one, or two (i.e., a finite number) of points. Since there is a countable number of S_k's, and since we showed on a previous homework that $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$, all the S_i's together only intersect *countably* many points on the circle. If the union of the S_i's were to be all of $\mathbb{R}^2$, it would surely cover every point in S^1, which we showed it does not. Hence $\mathbb{R}^2$ is not a *countable* union of subsets of straight lines in $\mathbb{R}^2$. □

Problem (4)**.** Prove that the set of real numbers can be written as the union of uncountably many pairwise disjoint subsets, each of which is uncountable.

Solution. Since $|\mathbb{R}^2| = |\mathbb{R}|$, there exists a bijection $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$. This induces the canonical equivalence relation on $\mathbb{R}$: $a \sim b \iff f(a) = f(b)$. Then consider the inverse map f from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Let $g(a) = \{x \in \mathbb{R} \mid \exists y \in \mathbb{R} \text{ s.t. } f(x) = (y, a)\}$. Then $f^{-1}(\mathbb{R}, a) = g(a)$. So $\mathbb{R} = \cup_{a\in\mathbb{R}} g(a)$, where all the $g(a)$'s are disjoint since f is a bijection, and each one is definitely uncountable (because of surjectivity), and we sum over uncountably many of them. □

Problem (5)**.** Let S be a subset of $\mathbb{R}^n$ with the distance function $d(x, y) = ((x_1 - y_1)^2 + \cdots + (x_n - y_n)^2)^{1/2}$ so that $(S, d\mid_{S\times S})$ is a metric space.

(a) Given $y \in S$, is $E = \{x \in S \mid d(x, y) \geq r\}$ a closed set in S?
(b) Is the set E in part (a) contained in the closure of $\{x \in S \mid d(x, y) > r\}$ in S?

Solution. (a). Yes. To show that E is closed in S, it suffices to show that $E^{\mathsf{c}} = \{x \in S \mid d(x, y) < r\}$ is open in S. Let's fix a point y. Consider the neighborhood $N_r(y) \in \mathbb{R}^n$; we know that $N_r(y)$ is an open subset of $\mathbb{R}^n$. Hence $E^{\mathsf{c}} = S \cap N_r(y)$. By **Theorem 2.30** in Rudin, E^{c} is open; hence, E is closed.

(b). □

Problem (6)**.** (a) Suppose that K and F are subsets of $\mathbb{R}^2$ with K closed and bounded and F closed. Prove that if $K \cap F = \varnothing$, then $d(K, F) > 0$. Recall that $d(K, F) = \inf\{d(x, y) \mid x \in K, y \in F\}$.
(b) Is (a) true if K is just closed? Prove your assertion.

Solution. (a). Assume for contradiction that $d(K, F) = 0$; in other words, $\inf\{d(x, y) \mid x \in K, y \in F\} = 0$. By the **Heine-Borel Theorem**, K is compact. Since $\inf\{d(x, y)\} = 0$, we can find some sequence $\{x_i\}$ where each $x_n \in K$, $\{y_n\}$ where each $y_n \in F$, such that $d(x_i, y_i)$ converges to 0. Since $\{x_n\} \subseteq K$ and K is compact, we can find some subset $F \subseteq I_n$ such that $\{x_i\}_{i\in F}$ converges to some $x^* \in K$. Consider this x^*. Since $d(x_i, y_i) = 0$, then $\forall \epsilon > 0$, $\exists i \in \mathbb{N}$ s.t. $d(x_i, y_i) < \epsilon$; by compactness, this becomes $d(x^*, y_i) < \epsilon$. Since each $y_i \in F$, x^* is a limit point of F. Since F is closed, $x^* \in F$. But $x^* \in K$ by compactness. Hence $K \cap F \neq \varnothing$, a contradiction. got 'em!

(b). This is false. Consider the closed sets $F = \{(n, 1/n) \mid n \geq 1, n \in \mathbb{N}\}$, and $K = \{(n, 0) \mid n \geq 1, n \in \mathbb{N}\}$[1]. The distance between K and F can get arbitrarily small, but F and K will never intersect. $\square$

Problem (7)**.** Let X be a complete metric space with metric d. Let $Y \subseteq X$. Prove that Y is closed in X iff Y with the metric $d \restriction Y$ is complete.

Solution. ($\Longrightarrow$): suppose that Y is closed in X. We want to show that $(Y, d \restriction Y)$ is complete. Consider some Cauchy sequence $\{y_n\}_{n\in\mathbb{N}}$. We have two cases:

Case I (y_n eventually becomes constant for all $n > m$ for some $m \in \mathbb{N}$): since it eventually becomes constant, $\{y_n\}$ converges to some y^*, but since that y^* is an element of the sequence, $y^* \in Y$, so Y is complete.

Case II (y_n does not become constant): since it never becomes constant but still converges, y_n must converge to a limit point y^* of Y (since we can always find a point in $\{y_n\}$ that's arbitrarily close to what it converges to, and $y_i \in Y$ $\forall i$ by the completeness of X). Since Y is closed, $y^* \in Y$; thus Y is complete.

($\Longleftarrow$): Suppose Y is complete under the metric d restricted to it. Take some limit point $y^* \in Y$. By **Theorem 3.2(d)** in Rudin, there is a sequence $\{y_n\}$ in Y that converges to y^*. Since Y is complete, $y^* \in Y$. Hence Y is closed. $\square$

Problem (8.R3.21)**.** Prove the following analogue of **Theorem 3.10(b)**: If $\{E_n\}$ is a sequence of closed and bounded sets in a *complete* metric space X, if $E_n \supseteq E_{n+1}$, and if

$$\lim_{n\to\infty} \operatorname{diam} E_n = 0,$$

then $\cap_1^\infty E_n$ consists of exactly one point.

Solution. Since the intersection is an intersection of nonempty closed sets, the intersection is nonempty so x^* can exist. Now suppose $\cap_1^\infty E_n$ has two points in it. Then for any arbitrarily large n, $\operatorname{diam} E_n > 0$. Then the limit does not converge, a contradiction. Hence the intersection contains exactly one point. $\square$

Problem (8.R3.22)**.** Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\cap_1^\infty G_n$ is not empty. (In fact, it is dense in X.) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $E_n \subseteq G_n$, and apply Exercise 21.

[1]They are definitely closed—consider the complement; it is $\mathbb{R}^2$ with some points removed, but the points are all distance > 1 apart, so it's open.

Solution. Let's find this sequence. Choose some $x_1 \in G_1$. Since G_1 is open, we can find some $\epsilon > 0$ such that $N_\epsilon(x) \subseteq G_1$. In addition, $N_{\epsilon/2}(x_1) \subsetneq N_\epsilon(x_1) \subseteq G_1$; moreover, by definition of limit points, we have that $N_{\epsilon/2}(x_1) \subsetneq \overline{N_{\epsilon/2}(x_1)} \subsetneq G_1$. Now consider G_2. Since its closure is all of X and $x_1 \in G_1 \subseteq X$, x_1 is in G_2 (or is a limit point). By Rudin, we can find a $\delta > 0$ where $\delta < \epsilon$ such that $x_2 \in N_\delta(x_1)$. In particular, $x_2 \in N_{\epsilon/2}(x_1)$. Similarly construct the neighborhood $N_{\delta/2}(x_2)$. We thus get the chain

$$\ldots \subsetneq N_i(x_i) \subsetneq \overline{N_i(x_i)} \subsetneq \ldots \subsetneq N_\gamma(x_3) \subsetneq \overline{N_{\gamma/2}(x_3)} \subsetneq N_{\delta/2}(x_2) \subsetneq \overline{N_{\delta/2}(x_2)} \subsetneq N_{\epsilon/2}(x_1) \subsetneq \overline{N_{\epsilon/2}}(x_1)$$

Considering just the chain of closed sets, (which are all bounded), we know from the previous exercise that their intersection contains exactly one point. But if there's exactly one point in each of the neighborhoods, then that point must be in each G_i, and hence in their intersection. □

Problem (11)**.** Let Y be a complete *countable* metric space. Prove that there is a point $y \in Y$ such that $\{y\}$ is open.

Solution. Note that a stronger statement of the **Baire category theorem**: a non-empty complete metric space, or any of its subsets with nonempty interior, is not the countable union of nowhere-dense sets[2]. Since Y is countable, we can enumerate each point as y_n where $n \in I_n$. Consider the sets $\{y_n\}$ for all n—assume for contradiction that none of them are open. Hence they would all have nonempty interior (since they are closed), but $Y = \cup_{i=1}^{\infty}\{y_n\}$, a contradiction. got 'em! □

[2]Source: ¸