Analysis HW #9

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Problem (1)**.** If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Solution. Let B be the upper bound of $\{|b_n|\}$. Since $\sum a_n$ converges, $\forall \varepsilon > 0$, $\exists N$ such that

$$\left|\sum_{k=m}^{n} a_k\right| \leq \frac{\varepsilon}{B} \iff \qquad (\forall n \geq m \geq N.)$$

$$\iff \left|\sum_{k=m}^{n} a_k \cdot B\right| \leq \varepsilon \implies$$

$$\implies \left|\sum_{k=m}^{n} a_k \cdot b_k\right| \leq \varepsilon \qquad \text{(Since } B \text{ is an upper bound)}$$

which shows that $\sum a_n b_n$ converges. □

Problem (2)**.** Prove the *arithmetic and geometric means* inequality, $\sqrt{uw} \leq \frac{u+w}{2}$ (for u, $w \geq 0$), and use this to show that $e^{\frac{x+y}{2}} \leq \frac{e^x+e^y}{2}$ for all x, y.

Solution. We'll need a small claim first.

Claim 0.1. Let $a, b \geq 0$. Then $a \leq b$ if an only if $a^2 \leq b^2$.

Proof. This claim is true from properties of order on the reals, as proven in Tao and earlier homeworks. □

Now, we prove AM-GM:

$$\sqrt{uw} \leq \frac{u+w}{2} \iff$$

$$uw \leq \left(\frac{u+w}{2}\right)^2 \iff \qquad \text{(By **Claim** 0.1)}$$

$$uw \leq \frac{u^2+2uw+w^2}{2^2} \iff$$

$$\begin{aligned} uw &\leq \frac{u^2}{4} + \frac{w^2}{4} + \frac{uw}{2} \iff \\ 0 &\leq \frac{u^2}{4} - \frac{uw}{2} + \frac{w^2}{4} \iff \\ 0 &\leq \left(\frac{u}{2} - \frac{w}{2}\right)^2. \end{aligned}$$

The last statement is always true since squares are always nonnegative, so $\sqrt{uw} \leq \frac{u+w}{2}$.

Now let $u = e^x$ and $w = e^y$. Then we get that

$$\begin{aligned} e^{\frac{x+y}{2}} &= (e^{x+y})^{\frac{1}{2}} \\ &= \sqrt{e^{x+y}} \\ &= \sqrt{e^x e^y} \\ &= \sqrt{uw} \\ &\leq \frac{u+w}{2} && \text{(By AM-GM)} \\ &= \frac{e^x + e^y}{2} \end{aligned}$$

as desired. □

Problem (3)**.** Prove that the exponential function is convex, meaning that if $x \leq y$ and $t \in [0, 1]$ then $e^{tx+(1-t)y} \leq te^x + (1-t)e^y$.

Solution. Essentially, what we're going to do is we'll prove that at the midpoint between x and y, the exponential function is less than the line connecting the points (x, e^x) and (y, e^y); we'll continue doing that iteratively, and then we'll use the density of these midpoints in the interval $[x, y]$ to show it for all points in the interval.

Claim 0.2. For $t = \frac{1}{2}$, this holds.

Proof. We want to show that $e^{\frac{1}{2}x+(1-\frac{1}{2})y} \leq \frac{1}{2}e^x + (1-\frac{1}{2})e^y$. This is algebraically equivalent to showing that $e^{\frac{x+y}{2}} \leq \frac{e^x+e^y}{2}$, which is true by **Problem 2**. □

Claim 0.3. This is true for all t in the form $t = \frac{k}{2^n}$, where k, $n \in \mathbb{N}$.

Proof. Fix k; we prove this by induction on n.

Base case ($n = 0$): we want to show that this holds for $t = \frac{k}{2}$. However, it is given that $t \in [0, 1]$, so $k \in \{0, 1, 2\}$.

Case I ($k = 0$): in this case, $t = 0$, so we want to show that $e^y \leq e^y$, which is trivially true.

Case II ($k = 2$): in this case, $t = 1$, so we want to show that $e^x \leq e^x$, which is trivially true.

Case III ($k = 1$): in this case, $t = \frac{1}{2}$, which is true by **Claim 0.2**.

Inductive step: suppose this is true for $t = \frac{k}{2^n}$. We want to show that this is true for $t' = \frac{k}{2^{n+1}}$. Observe that t' is the midpoint of $\frac{(\frac{k-1}{2})}{2^n}$ and $\frac{(\frac{k+1}{2})}{2^n}$. So let $x' = \frac{(\frac{k-1}{2})}{2^n}$ and $y = \frac{(\frac{k+1}{2})}{2^n}$. Relative to x' and y', $t' = \frac{1}{2}$, so by **Claim 0.2**, the inequality holds at $t' = \frac{k}{2^{n+1}}$, which closes the induction. $\square$

Now, for any $s \in [0, 1]$, we can find some sequence $\{t_n\}$, where each t_i is of the form $\frac{k_i}{2^{n_i}}$, such that $s = \sup\{t_n\}$. Thus the inequality $e^{sx+(1-s)y} \leq se^x + (1-s)e^y$ holds for all points x, y, where $x \leq y$ and $s \in [0, 1]$, as desired. $\square$

Problem (4)**.** Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$.

Solution. We will follow the proof outline suggested in the book, which ultimately reduces to proving the following seven claims, which we shall do.

Claim 0.4. For any positive integer n, $b^n - 1 \geq n(b - 1)$.

Proof. We prove this by induction on n.

Base case ($n = 1$): we want to show that $b^1 - 1 \geq 1(b - 1)$. This is true since $b^1 - 1 = b - 1 = 1(b - 1)$.

Inductive step: suppose $b^n - 1 \geq n(b + 1)$. Then we get that

$$\begin{aligned}
b^{n+1} - 1 &= b^n \cdot b - 1 \\
&\geq (n(b-1) + 1) \cdot b - 1 && \text{(By the IH)} \\
&= (nb - n + 1) \cdot b - 1 \\
&= nb^2 - nb + b - 1 \\
&= nb(b-1) + (b-1) \\
&= (nb+1)(b-1) \\
&\geq (n+1)(b-1) && \text{(Since } b > 1\text{)}
\end{aligned}$$

which completes the induction. $\square$

Claim 0.5. $b - 1 \geq n(b^{1/n} - 1)$.

Proof. This is a direct consequence of **Claim 0.4**. $\square$

Claim 0.6. If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

Proof. Suppose $t > 1$. Then we have that

$$\begin{aligned}
n > (b-1)/(t-1) &\implies \\
n > n(b^{1/n}-1)/(t-1) &\implies && (\textbf{By Claim 0.5}) \\
1 > (b^{1/n}-1)/(t-1) &\implies && (\text{Since } n \geq 1) \\
t-1 > b^{1/n}-1 &\implies && (\text{Since } t-1 > 0) \\
t > b^{1/n}. &
\end{aligned}$$

□

Claim 0.7. If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n.

Proof. Let $t = y \cdot b^{-w}$. For n sufficiently large, we can get that $n > (b-1)/(t-1)$. Then since $t > 1$, using **Claim 0.6** we get that $b^{w+(1/n)} = b^w b^{1/n} < b^w t = b^w \cdot y \cdot b^{-w} = y$. □

Claim 0.8. If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.

Proof. Since $b^w > y$, $b^w/y > 1$. Let $t = b^w/y$. By **Claim 0.6**, for sufficiently large n, we get that $b^w y^{-1} > b^{1/n}$. Multiplying both sides by $b^{-1/n}y$, we get that $b^{w-(1/n)} > y$, as desired. □

Claim 0.9. Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

Proof. We now by trichotomy that exactly one of $b^x > y$, $b^x < y$, or $b^x = y$ is true.

Case I ($b^x > y$): by **Claim 0.8**, choose some n such that $b^x > b^{x-(1/n)} > y$. Thus $x - (1/n)$ is an upper bound of A, but it is smaller than x, which gives a contradiction. got 'em!

Case II ($b^x < y$): by **Claim 0.7**, choose some n such that $y > b^{x+(1/n)} > b^x$. But now x is not an upper bound for A, which gives a contradiction. got 'em!

Hence $b^x = y$ must be true. □

Claim 0.10. Prove that this x is unique.

Proof. This is true by the uniqueness of suprema. □

By **Claims 0.9** and **0.10**, we proved that the logarithm of y to the base b exists and is unique for reals $b > 1$, $y > 0$. □

Problem (5)**.** Determine for each of the following series whether or not it converges.

(a) $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$.
(b) $\sum_{n\geq 1} \frac{n!}{n^n}$.
(c) $\sum_{n=2}^{\infty} \frac{n^{\log_2 n}}{(\log_2 n)^n}$.
(d) $\sum_{n\geq 2} \frac{1}{(\log_2 n)^{\log_2 n}}$.
(e) $\sum_{n\geq 1} \frac{(-1)^n n!}{2^n}$.

Solution. (a). Observe that the terms of this series are either of the form $\frac{1}{(n-1)^2}$ or $\frac{1}{(n+1)^2}$. Hence,

$$\begin{aligned}\sum \frac{1}{[n+(-1)^n]^2} &\leq \sum \frac{1}{(n-1)^2} \\ &\leq \frac{1}{n^{3/2}}\end{aligned}$$

where the last series converges by the p-series test. Hence, $\sum_{n=2}^{\infty} \frac{1}{[n+(-1)^n]^2}$ converges.

(b). Observe that

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(n+1)!}{(n+1)^{(n+1)}} \cdot \frac{n^n}{n!} \right| &= \lim_{n\to\infty} \left| \frac{(n+1)n^n}{(n+1)^{(n+1)}} \right| \\ &= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right| \\ &= e^{\lim_{n\to\infty} n\log n - n\log(n+1)} \\ &= e^{-1} = \frac{1}{e} < 1.\end{aligned}$$

Hence, by the ratio test, the sequence converges absolutely, since $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

(c). By the Cauchy condensation test, the series converges if and only if the series $\sum_{n=2}^{\infty} 2^n \frac{(2^n)^{\log_2(2^n)}}{(\log_2(2^n))^{(2^n)}}$. We get that the latter series is equivalent to

$$\begin{aligned}\sum_{n=2}^{\infty} 2^n \frac{(2^n)^{\log_2(2^n)}}{(\log_2(2^n))^{(2^n)}} &= \sum_{n=2}^{\infty} 2^n \frac{(2^n)^n}{(n)^{(2^n)}} \\ &= \sum_{n=2}^{\infty} \frac{(2^n)^{n+1}}{n^{2^n}}.\end{aligned}$$

Applying the ratio test to this series, we get

$$\lim_{n\to\infty} \frac{\frac{(2^{(n+1)})^{n+2}}{(n+1)^{2^{(n+1)}}}}{\frac{(2^n)^{n+1}}{n^{2^n}}} = \lim_{n\to\infty} 4^{n+1} n^{2^n} (n+1)^{-2^{n+1}}$$
$$= 0$$

Hence, the larger series converges. By the comparison test, our series converges as well.

(d). We use the Cauchy condensation test once again—our series converges if and only if the series $\sum_{n=2}^{\infty} 2^n \frac{1}{(\log_2 2^n)^{(\log_2 2^n)}}$ converges. The latter series is simply $\sum_{n=2}^{\infty} \frac{2^n}{n^n}$. But for all $n \geq 4$, $2 \leq n/2$, so $\frac{2^n}{n^n} \leq \frac{(n/2)^n}{n^n} = \frac{1}{2^n}$, so by the comparison test, the latter series converges; thus, our series converges as well.

(e). Observe that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|(-1)\frac{(n+1)!}{2^{n+1}}\frac{2^n}{n!}\right|$$
$$= \lim_{n\to\infty} \left|\frac{(n+1)}{2}\right| = \infty > 1$$

so by the ratio test, this series diverges. □

Problem (6)**.** Let $\{a_n\}_{n\geq 1}$ be a decreasing sequence of non-negative numbers. Suppose $\sum_{n\geq 1} a_n$ converges. Prove that $\lim_{n\to\infty} na_n = 0$.

Solution. Since $\{a_n\}$ is positive and nonincreasing, we can apply the Cauchy condensation test: that is, $\sum_{n\geq 1} a_n$ converges if and only if $\sum_{n\geq 1} 2^n a_{2^n}$ converges. To show the desired result, we will prove the contrapositive of the statement: namely, if $\lim_{n\to\infty} na_n \neq 0$, then $\sum_{n\geq 1} 2^n a_{2^n}$ diverges. So suppose $\lim_{n\to\infty} na_n \neq 0$. For contradiction, suppose $\sum_{n\geq 1} 2^n a_{2^n}$ converges. In a convergent series $\sum x_n$, $\{x_n\}$ must converge to 0. Hence $\lim_{n\to\infty} 2^n a_{2^n} = 0$. But this is a contradiction, since we said $\lim_{n\to\infty} na_n \neq 0$, which clearly fails in the case where we look at the terms in positions of powers of two. □

Problem (7)**.** Prove that $\sum_{n\geq 1} \frac{1}{n(n+1)} = 1$.

Solution. Suffices to show that the sequence of partial sums $\{s_n\}$ converges to 1, where $s_n = \sum_{n=1}^{k} \frac{1}{n(n+1)}$. We showed in class that for any monotone increasing sequence that is bounded above, its supremum is the limit. Thus, we will show several things:

Claim 0.11. $\{s_n\}$ is monotone increasing.

Proof. Observe that $s_{n+1} = a_{n+1} + s_n = \frac{1}{(n+1)(n+2)} + s_n$. Since $\frac{1}{(n+1)(n+2)} > 0$, $s_{n+1} > s_n$. $\square$

Claim 0.12. $\{s_n\}$ is bounded above.

Proof. Suppose not. Then the sequence of partial sums of a_n diverges, so $\sum a_n$ diverges, so $\sum_{n\geq 1} \frac{1}{n(n+1)}$ diverges. However, this is a contradiction, since $\sum a_n$ converges by a comparison with the series $\sum \frac{1}{n^2}$. $\square$

Claim 0.13. $\sup s_n = 1$.

Proof. We need to show two things:

1. 1 is an upper bound of $\{s_n\}$; and
2. 1 is the least upper bound of $\{s_n\}$.

Proof of 1: Observe that $s_n = \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \frac{n}{n+1}$. Suppose 1 were not an upper bound of $\{s_n\}$. Then we can find some k such that $s_k > 1$. But this implies that $\frac{k}{k+1} > 1 \iff k > k+1$, which is a contradiction. got 'em!

Proof of 2: Suppose $\exists z < 1$ that is an upper bound for $\{s_n\}$. Choose some $r \in \mathbb{N}$ such that $r > \frac{z}{1-z}$, which exists by the Archimedean property. Then $s_r = \frac{r}{r+1} > \frac{\frac{z}{1-z}}{1-\frac{z}{1-z}} = \frac{\frac{z}{1-z}}{\frac{1-z}{1-z} - \frac{z}{1-z}} = \frac{\frac{z}{1-z}}{\frac{1}{1-z}} = z$, a contradiction. got 'em! $\square$

By the three claims above, the partial sums of $\sum_{n\geq 1} \frac{1}{n(n+1)}$ converge to 1, so the series converges to 1 as well. $\square$

Problem (8)**.** Prove that $\sum_{n\geq 1} \frac{n-1}{2^{n+1}} = \frac{1}{2}$, and use this to calculate $\sum_{n\geq 1} \frac{n}{2^n}$.

Lemma 1

$\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = 1$.

Proof. Let $s_k = \sum_{n=1}^{k} \frac{n}{2^{n+1}}$. We claim that $s_k = 2^{-k-1}(-k + 2^{k+1} - 2)$. We show this by induction:

Base case ($k = 1$):

$$\begin{aligned} \sum_{n=1}^{k} \frac{n}{2^{n+1}} &= \sum_{n=1}^{1} \frac{n}{2^{n+1}} \\ &= \frac{1}{2^{1+1}} \\ &= \frac{1}{4} \end{aligned}$$

$$
\begin{aligned}
&= 2^{-2} \\
&= 2^{-2}(1) \\
&= 2^{-1-1}(1) \\
&= 2^{-1-1}(-1+4-2) \\
&= 2^{-1-1}(-1+2^2-2) \\
&= 2^{-1-1}(1+2^{1+1}-2) \\
&= 2^{-1-1}(-1+2^{1+1}-2) \\
&= 2^{-k-1}(-k+2^{k+1}-2).
\end{aligned}
$$

Inductive step: suppose that $s_k = 2^{-k-1}(-k+2^{k+1}-2)$. Then

$$
\begin{aligned}
s_{k+1} &= s_k + a_{k+1} \\
&= 2^{-k-1}(-k+2^{k+1}-2) + a_{k+1} \\
&= 2^{-k-1}(-k+2^{k+1}-2) + \frac{k+1}{2^{k+2}} \\
&= 2^{-k-1}(-k+2^{k+1}-2) + 2^{-k-2}(k+1) \\
&= 2^{-k-1}(-k+2^{k+1}-2) + 2^{-k-1}(2^{-1}(k+1)) \\
&= 2^{-k-1}(-k+2^{k+1}-2+2^{-1}k+2^{-1}) \\
&= 2^{-k-2}(-2k+2^{k+2}-4+k+1) \\
&= 2^{-k-2}(-k+2^{k+2}-3) \\
&= 2^{-(k+1)-1}(-k+2^{k+2}-3) \\
&= 2^{-(k+1)-1}(-k-1+2^{k+2}-2) \\
&= 2^{-(k+1)-1}(-(k+1)+2^{k+2}-2) \\
&= 2^{-(k+1)-1}(-(k+1)+2^{(k+1)+1}-2)
\end{aligned}
$$

which closes the induction.

Hence, $\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{n}{2^{n+1}} = \lim_{k\to\infty} 2^{-(k+1)-1}(-(k+1)+2^{(k+1)+1}-2) = 1$. □

Lemma 2

$\sum_{n=1}^{\infty} \left(\frac{1}{2^{n+1}}\right) = \frac{1}{2}$.

Proof. Let $s_k = \sum_{n=1}^{k} \left(\frac{1}{2^{n+1}}\right)$. We claim that $s_k = \frac{2^k-1}{2^{k+1}}$.

We show this by induction on k.

Base case ($k=1$):

$$\begin{aligned}\sum_{n=1}^{k}\left(\frac{1}{2^{n+1}}\right) &= \sum_{n=1}^{1}\left(\frac{1}{2^{n+1}}\right)\\ &= \frac{1}{2^{1+1}}\\ &= \frac{2-1}{2^{k+1}}\\ &= \frac{2^1-1}{2^{k+1}}\\ &= \frac{2^k-1}{2^{k+1}}.\end{aligned}$$

Inductive case: suppose that $s_k = \frac{2^k-1}{2^{k+1}}$. Then we have that

$$\begin{aligned}s_{k+1} = s_k + a_{k+1}&\\ &= \frac{2^k-1}{2^{k+1}} + \frac{1}{2^{(k+1)+1}}\\ &= \frac{2^k-1}{2^{k+1}} + \frac{1/2}{2^{k+1}}\\ &= \frac{2^k-1+1/2}{2^{k+1}}\\ &= \frac{2^k-1/2}{2^{k+1}}\\ &= \frac{2(2^k-1/2)}{2(2^{k+1})}\\ &= \frac{2^{k+1}-1}{2^{(k+1)+1}}\end{aligned}$$

which closes the induction.

Hence $\sum_{n=1}^{\infty}\frac{1}{2^{n+1}} = \lim_{k\to\infty}\sum_{n=1}^{k}\frac{1}{2^{n+1}} = \lim_{k\to\infty}\frac{2^{k+1}-1}{2^{(k+1)+1}} = 1/2.$ □

Solution. Observe that $\sum_{n=1}^{k}\frac{n-1}{2^{n+1}} = \sum_{n=1}^{k}(\frac{n}{2^n} - \frac{n+1}{2^{n+1}})$, which conveniently telescopes to $\frac{1}{2} - \frac{k+1}{2^{k+1}}$. Letting k go to infinity, we get that $\sum_{n=1}^{\infty}\frac{n-1}{2^{n+1}} = \lim_{k\to\infty}\sum_{n=1}^{k}\frac{n-1}{2^{n+1}} = \lim_{k\to\infty}\left(\frac{1}{2} - \frac{k+1}{2^{k+1}}\right) = \frac{1}{2}$. Now we know that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{n}{2^n}\right) &= \sum_{n=1}^{\infty}\left(\frac{n-1}{2^{n+1}} + \frac{n+1}{2^{n+1}}\right)\\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{n+1}{2^{n+1}}\right)\\ &= \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{n}{2^{n+1}}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{2^{n+1}}\right)\\ &= \frac{1}{2} + 1 + \sum_{n=1}^{\infty}\left(\frac{1}{2^{n+1}}\right) && \text{(By \textbf{Claim 1})}\end{aligned}$$

$$= \frac{1}{2} + 1 + \frac{1}{2} \qquad \text{(By \textbf{Claim} 2)}$$
$$= 2.$$

□

Problem (9)**.** A metric space is called *sequentially compact*, or said to have the *Bolzano-Weierstraß property*, if every sequence has a convergent subsequence. A metric space is *totally bounded* if for every $\varepsilon > 0$ the space can be covered by finitely many open balls of radius ε. Prove that the following are equivalent:

(a) X is compact.
(b) X is sequentially compact.
(c) X is complete and totally bounded.

Solution. (a $\implies$ c): suppose X is compact. Consider the following open cover: $G = \cup_{x \in X} N_\varepsilon(x)$. It is definitely open, since each neighborhood is open, and the union of open sets is open; it is a cover, since each point is, at the very least, in the neighborhood centered around it. Since X is compact, there exists a finite subcover of G. Hence, X can be covered by finitely many open balls of radius ε. But since our choice of an open cover was independent of our choice of $\varepsilon > 0$, this is true $\forall \varepsilon > 0$; hence X is totally bounded. Moreover, compactness is a stronger condition on metric spaces than completeness, so X is complete[1].

(a $\implies$ b): suppose X is compact. Let $\{x_n\}$ be a sequence of points in X. We want to show that $\{x_n\}$ has a convergent subsequence. To do this, let $E = \{x_n \mid n \in \mathbb{N}\}$. We have two natural cases—either E is finite or infinite.

Case I (E is finite): in this case, then $\exists x^* \in E$ repeated infinitely many times. So take the subsequence of repetitions of x^*, and we get the obviously convergent subsequence $\{n_k\}_{k=1}^{\infty}$, where $x_{n_k} = x^*$.

Case II (E is infinite): If E is infinite, then E has a limit point x^* (by the compactness of X). Since x^* is a limit point, define n_k by recursion on k as follows:

(1) set $n_1 = 1$;

(2) assuming n_k has been defined, since x^* is a limit point of E, there are infinitely many points in E within distance $\frac{1}{k+1}$ of x^*. In particular, we can find

[1] Obviously, if every sequence in X has a convergent subsequence that converges to a point in X, then every *Cauchy* sequence in X has a convergent subsequence that converges to a point in X; but if a subsequence of a Cauchy sequence converges to X, then the Cauchy sequence converges to X as well. Hence every Cauchy sequence converges to a point in X, so X is complete.

some $x_m \in E$ such that $d(x_m, x^*) < \frac{1}{k+1}$ and $m > n_k$ (since there are infinitely many, and n_k is finite). So set n_{k+1} equal to such an m.

Then $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, and $(\forall k > 1)$ $d(x^*, x_{n_k}) < \frac{1}{k}$. Hence $\{x_{n_k}\}$ converges to x^*.

Thus X is sequentially compact.

(c $\implies$ a): suppose X is complete and totally bounded. We want to show that X is compact—that is, every sequence $\{x_n\}$ in X has a subsequence that converges to a point in X. So let's take some sequence $\{x_n\}$ in X. Since X is complete, every Cauchy sequence converges to a point in X; hence, suffices to find a subsequence of $\{x_n\}$ that is Cauchy. Let $E = \{x_n \mid n \in \mathbb{N}\}$. We have two cases about the finiteness of E:

Case I (E is finite): in this case, some point in $\{x_n\}$ must repeat infinitely many times. Thus, take the subsequence of $\{x_n\}$ that is just that point repeated over and over. Clearly it is Cauchy, so we're done.

Case II (E is infinite): observe that since X is totally bounded, the space can be covered by finitely many open balls of radius ε. Consider the covering by the minimal number of open balls for any given ε. Since E is infinite and there are *finitely* many open balls, we can find an open ball $N_\varepsilon(x)$ of radius ε that contains infinitely many points in E by the infinite pigeonhole principle. Let $\{b_n\}_\varepsilon$ be a subsequence of $\{a_n\}$ that contains just the infinitely many points in $N_\varepsilon(x)$. We claim that $\{b_n\}_\varepsilon$ converges to x. Well, $\forall \varepsilon > 0$, the maximum distance between x and any b_i is less than ε by the definition of an open ball. Hence $\{b_n\}_\varepsilon$ converges. But x is in the neighborhood surrounding it, and since ε can be arbitrarily small, and the finite open cover is minimal, x must be in X.

Thus, every sequence in X has a subsequence that converges to a point in X.

(b $\implies$ c): suppose X is sequentially compact. First, let's show that this implies that X is complete. Take some Cauchy sequence $\{x_n\}$ in X. We want to show that it converges in X. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. Let x^* be a limit point of $\{x_{n_k}\}$. Fix $\varepsilon > 0$. We want to show that $\{x_n\}$ converges to the same limit as $\{x_{n_k}\}$; hence, we need some $N \in \mathbb{N}$ such that $(\forall n \geq N)$, $d(x_n, x^*) < \varepsilon$.

Since $\{x_{n_k}\}$ converges to x^*, we have an N_1 such that $\forall k \geq N_1$, $d(x_{n_k}, x^*) <$ $\varepsilon/2$. Since $\{x_n\}$ is Cauchy, we have some N_2 such that $\forall n, m \geq N_2$, $d(x_n, x_m) <$

$\varepsilon/2$. Take $N = \max(N_1, N_2)$[2]. Fix $n \geq N \geq N_2$. We will show that $d(x_n, x^*) < \varepsilon$.

Let k be large enough that $k \geq N_1$ and $n_k \geq N_2$. Then $d(x_{n_k}, x^*) < \varepsilon/2$. But since $k, n_k \geq N_2$, $d(x_{n_k}, x_n) < \varepsilon/2$, so by the triangle inequality, $d(x_n, x^*) < \varepsilon$. Hence X is complete.

Now, we want to show that X is totally bounded—that is, for every $\varepsilon > 0$, X can be covered by finitely many open balls of radius ε. Suppose it isn't. Then $\exists \varepsilon > 0$ for which there is no finite covering of X with open balls of size ε. So construct a sequence as follows: let $x_1 \in X$. Since $N_\varepsilon(x_1) \subsetneq X$[3], we can find some x_2 such that $d(x_1, x_2) > \varepsilon$. Recursively, given we already have the points $\{x_1, x_2, \ldots, x_n\}$, choose the point x_{n+1} to be such that $d(x_i, x_{n+1}) > \varepsilon \ \forall i$. We know such a point exists, for if it didn't, then the n neighborhoods of radius ε around then x_i's would form a finite cover of X. But the sequence $\{x_n\}$ is very not Cauchy, so it has no convergent subsequences. But then X is not sequentially compact, a contradiction. got 'em! $\square$

[2] In fact, $N = N_2$ works.

[3] Necessarily proper—otherwise we would have a finite cover of open balls with radius ε, so the space would be totally bounded, which we're assuming it isn't.