Analysis HW #4

Dyusha Gritsevskiy

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Problem (1)**.** Let A be a non-empty set of integers, bounded above by $n \in \mathbb{N}$ and bounded below by $-n$. Prove that A has a largest element. Use this to show that any non-empty set of integers that is bounded above, has a largest element.

Solution. Suppose that A has no largest element. In other words, if we fix any $x \in A$, we can find some $u \in A$ such that $u > x$. Since A is a nonempty set that is bounded above, A has a least upper bound, which we will call b. We know that $b \notin A$, for if it was, then $b \geq x$ for any $x \in A$, so A has a largest element, a contradiction. By the Archimedean principle, $\bar{b}, b^* \in \mathbb{Z}$ s.t. $\bar{b} < b < b^*$. Let $\bar{b}$ be the largest integer satisfying this inequality, and b^* be the smallest integer satistying this inequality. Consider $\bar{b}$. If $\bar{b} \in A$, then we can find some $u \in A$ such that $u > \bar{b}$. Since b is an upper bound for A, $u \leq b$. Thus $\bar{b} < u \leq b$. Since $u \in A$, u is an integer which is less than b, so $\bar{b}$ is no longer the largest integer satisfying the inequality, a contradiction. Thus, it must be the case that $\bar{b} \notin A$. We claim that $\bar{b}$ is an upper bound for A. Suppose not. Then there exists some $s \in A$ such that $s > \bar{b}$. Since $s \in A$, s is an integer, and $\bar{b} < s \leq b$, since b is an upper bound. Hence, $\bar{b}$ is no longer the largest integer satisfying the inequality, a contradiction once again. We arrive at a point where $\bar{b}$ can be neither in A or not in A; hence, our original assumption that A has no largest element must have been wrong, which completes the proof. □

Problem (2)**.** Let x, y be positive rationals. Prove that $\sup\{p \cdot q \mid p, q \in \mathbb{Q}, p, q \geq 0, p < x, q < y\} = x \cdot y$.

Solution. Essentially, we want to show that multiplication on the reals agrees with multiplication on the rationals. To do this, we need to show two things:

1. $p \cdot q \leq x \cdot y$ if $p < x$ and $q < y$ for positive rationals p, q; and
2. $x \cdot y$ is the smallest number for which (1) holds.

Proof of ① : Since we're operating on rationals, we can use our usual rational definition of order. Hence, $x = p + c$ and $y = q + d$ for positive rationals c, d. Hence, $x \cdot y = (p + c) \cdot (q + d) = pq + pd + cq + cd$, since the rationals are an ordered field. Since p, q, c, and d are all positive rationals, $pd + cq + cd$ is also a positive rational. Hence, $x \cdot y = pq + (pd + cq + cd)$, so $p \cdot q \leq x \cdot y$.

Proof of ② : Fix some $z < x \cdot y$. We want to show that z is *not* an upper bound for $\{p \cdot q \mid p, q \in \mathbb{Q}, p, q \geq 0, p < x, q < y\}$. To be safe, fix some $s \in Q$ such that $z < s < x \cdot y$, so we continue working on rationals; suffices to show that s is not an upper bound. Therefore, it suffices to find some p, q such that their product is greater than or equal to s. Since we're working on rationals, this is pretty easy. Let $d = xy - s$. Since $s < xy$, d is a positive rational. Now take a positive rational $f = x + y$, and let $p = x - \frac{d}{f}$ and $q = y - \frac{d}{f}$. Clearly, $\frac{d}{f}$ is positive, so $p < x$ and $q < y$, as needed. Now, we get that

$$p \cdot q = \left(x - \frac{d}{f}\right)\left(y - \frac{d}{f}\right) = xy + \frac{d^2}{f^2} - \frac{dx}{f} - \frac{dy}{f} \tag{1}$$

By properties of order on rationals, $pq < xy$; hence, suffices to show that $pq > s$. But we have that

$$\begin{aligned} pq > s &\iff \\ xy + \frac{d^2}{f^2} - \frac{dx}{f} - \frac{dy}{f} > xy - d &\iff \\ \frac{dx}{f} + \frac{dy}{f} - \frac{d^2}{f^2} - d < 0 &\iff \\ d + \frac{d^2}{f^2} > \frac{dx}{f} + \frac{dy}{f} &\iff \\ 1 + \frac{d}{f^2} > \frac{x}{f} + \frac{y}{f} &\iff && \text{(Since } d > 0\text{)} \\ 1 + \frac{d}{f^2} > \frac{x+y}{f} &\iff \\ 1 + \frac{d}{f^2} > \frac{x+y}{x+y} &\iff \\ \frac{d}{(x+y)^2} > 0 & \end{aligned}$$

which is definitely true, since d is positive and $(x + y)^2$ is a positive rational.

□

Problem (3)**.** Let x be a positive rational. Prove that $\inf\{1/p \mid p \in \mathbb{Q}, p > 0, p < x\} = 1/x$.

Solution. We do this in a way similar to problem 2. Essentially, we need to show two things:

1. $1/p \geq 1/x$ if $p < x$ for some positive rational p; and
2. $1/x$ is the largest number for which ⓵ holds.

Proof of ⓵: Since p and x are both rationals, this follows from properties of order.

Proof of ⓶: Fix some $z > 1/x$. We want to show that z is *not* a lower bound for $\{1/p \mid p \in \mathbb{Q}, p > 0, p < x\}$. To be safe, fix some $s \in Q$ such that $1/x < s < z$, so we continue working on rationals; suffices to show that s is not a lower bound. Thus, suffices to find some $p < x$ such that $1/p < s$. Let $d = s - 1/x$; since $s > 1/x$, d is a positive rational. Let's choose $p = x - \frac{sx-1}{2s}$. As $s > 1/x$, $sx > 1$, so $sx - 1 > 0$, so $p < x$, as needed. Now, suffices to show that $1/p < s$, which we do as follows (to simplify things, we let $f = \frac{2s}{x}$ and $d = s - \frac{1}{x}$):

$$\begin{aligned} \frac{1}{p} < s &\iff \\ \frac{f}{xf - d} < s &\iff \\ f(1 - sx) < -sd &\iff \\ f > \frac{-sd}{1 - sx} &\iff && \text{(Since } s > 1/x\text{)} \\ f > \frac{sd}{sx - 1} &\iff \\ f > \frac{s}{x} &\iff \\ \frac{2s}{x} > sx &\iff \\ 2 > 1 & && \text{(Since } s > 0 \text{ and } x > 0\text{)} \end{aligned}$$

which is always true. □

Problem (4)**.** Let x be a positive real. Prove that $x \cdot (\frac{1}{x}) = 1$.

Solution. Have to show that

$$\sup\{p \cdot q \mid p, q \in \mathbb{Q}, p, q \geq 0, p < x, q < \inf\{1/p \mid p \in \mathbb{Q}, p > 0, p < x\}\} = 1 \quad (2)$$

in other words,

$$\sup\{p \cdot q \mid p, q \in \mathbb{Q}, p, q \geq 0, p < x, q < 1/x\} = 1 \quad (3)$$

To do this, we need to demonstrate two things:

1. if $p, q \in \mathbb{Q}$, $p < x$, $q < 1/x$, $p > 0$, $q > 0$, then $p \cdot \left(\frac{1}{q}\right) \leq 1$; and
2. 1 is the least real for which (1) holds.

Proof of 1: Fix p, q. Since $q < 1/x = \inf\{1/r \mid r \in \mathbb{Q}, r > 0, r < x\}$, q is a lower bound for $\inf\{1/r \mid r \in \mathbb{Q}, r > 0, r < x\}$. Since $p < x$, $1/p > 1/x$[1]; since q is the greatest lower bound, $1/p \in \{1/r \mid r \in \mathbb{Q}, r > 0, r < x\}$. Hence $q \leq 1/p$, so $pq \leq 1$.

Proof of 2: Fix some $z < 1$. Want to show that z is *not* an upper bound for $\{p \cdot q \mid p, q \in \mathbb{Q}, p, q \geq 0, p < x, q < 1/x\}$. First, for convenience, fix some $s \in \mathbb{Q}$ s.t. $z < s < 1$. Thus, suffices to show that s is not an upper bound for the set. Hence, suffices to find some p, q such that the product of the former and the reciprocal of the latter is greater than s. Fix some $p < x$. □

Problem (5)**.** For all reals $b, s > 1$, prove that there is an $n \in \mathbb{N}$ so that $b^{1/n} < s$.

Lemma 1

For all $\epsilon > 0$, $n \in N$, $(1+\epsilon)^n \geq 1 + n\epsilon$.

Proof. We prove this by induction on n.

Base case ($n = 0$): $(1+\epsilon)^n = 1 \geq 1 + 0 = 1 + 0\epsilon = 1 + n\epsilon$.

Inductive step: suppose that $(1+\epsilon)^n \geq 1+n\epsilon$. Then $(1+\epsilon)^{n+1} = (1+\epsilon)^n(1+\epsilon) \geq (1+n\epsilon)(1+\epsilon) = 1 + n\epsilon^2 + \epsilon(1+n) = 1 + (n+1)\epsilon + n\epsilon^2 \geq 1 + \epsilon(1+n)$, where the second inequality used the inductive hypothesis and the last step used the fact that $n\epsilon^2 > 0$, which completes the induction. □

Solution. Since $s = 1$, let $s = 1 + \epsilon$ for $\epsilon > 0$. Suffices to find some $n \in \mathbb{N}$ such that $s^n > b$. But iff $s^n > b$, then $(1+\epsilon)^n > b$, so $1 + n\epsilon > b$ so $n\epsilon > b - 1$ so $n > \frac{b-1}{\epsilon}$. Such an n exists by the Archimedean property. □

Problem (6)**.**

Solution. (a). Let $x = mq = np$, and $y = nq$. Since the y^{th} root of b^k number is unique, it suffices to show that $((b^m)^{\frac{1}{n}})^y = ((b^p)^{\frac{1}{q}})^y$. The left-hand side of the equation simply equals $((b^m)^{\frac{1}{n}})^{nq} = (b^m)^q = b^{mq} = b^k$; the right-hand side equals $((b^p)^{\frac{1}{q}})^{nq} = (b^p)^n = b^{pn} = b^k$, so we are done. □

Problem (7)**.** Recall that $I_n = \{0, 1, \ldots, n-1\}$ for a natural number n. Prove the pigeonhole principle: for all $n \in \mathbb{N}$, every 1-to-1 function $f : I_n \to I_n$ is onto.

[1] Follows via density and order on rationals.

Solution. We prove this using induction.

Base case ($n = 0$): consider any 1-to-1 function from $\{0\}$ to $\{0\}$; since the target has exactly one element, every function maps to 0. Since 0 is the only element in the target, every function is into, including injective functions.

Inductive step: suppose that every 1-to-1 function $f : I_n \to I_n$ is onto. We want to show that every 1-to-1 function $f : I_{n+1} \to I_{n+1}$ is onto. Consider some 1-to-1 function $a : I_{n+1} \to I_{n+1}$. Consider how it acts on the subset $I_n \subset I_{n+1}$; since it is one-to-one, it acts on a set of n elements, so by the inductive hypothesis it is surjective. Hence, the restriction of a on I_n is a bijection. Now consider how $a(n-1)$. If it maps to any elements in the codomain of the restriction of a, a is no longer one-to-one; hence, it must map to the remaining unmapped-to element in its target. Hence, every element in the target is mapped to, so a is a surjection. □

Problem (8)**.** Suppose A is finite, and $B \subsetneq A$. Prove that $B \not\approx A$.

Solution. We prove this via strong induction on $|A|$.

Base case ($|A| = 1$): A contains one element—let's call this element x. Since $B \subset A$, either $B = \{x\}$ or $B = \{\}$. In the former case, $B = A$, which is not allowed. Hence, $B = \{\}$. There is a bijection between A and I_n; specifically, $f : A \to I_n$ given by $x \mapsto 0$. However, there is no bijection between B and I_n; if that were the case, $f(0)$ in I_n would have to map to something in B, but there is nothing to map to, a contradiction.

Inductive step: suppose that for any $B \subsetneq A$, $B \not\approx A$, where $|A| = n$. Suppose $|A| = n++$. Since $B \subsetneq A$, we have two cases: either $|B| < n$, or $|B| = n$[2]. In the former case, assume there is some bijection $f : B \to A$. Since f is a bijection, it maps to $|B|$ elements in A; hence, its codomain is a subset of A. But $B \subsetneq A$, so by the inductive hypothesis, such a bijection cannot exist. In the latter case, there is no bijection between A and B by the inductive hypothesis. This completes the induction. □

Problem (9)**.** Rudin 2.4: Is the set of all irrational real numbers countable?

Lemma 2

The union of two countable sets is countable.

[2]If $|B| = n + 1 = |A|$, then there is a bijection between A and B; since B is a subset of A, $A = B$, a contradiction. If $|B| > n + 1$, then B is no longer a subset of A by various set-theoretic counting theorems.

Proof. Consider two countable sets A_1, A_2. Since they are countable, there is a bijection $f : \mathbb{N} \to A_1$, and a bijection $g : N \to A_2$. Consider the set $(f(0), g(0), f(1), g(1), \ldots)$. This set definitely maps to all of $A_1 \cup A_2$, and there exists a bijection from $\mathbb{N}$ to this set (the obvious bijection, where 0 maps to $f(0)$, 1 maps to $g(0)$, n maps to either f or g or $n/2$ or $(n+1)/2$ depending on whether n is even or odd). Since the composition of bijections is a bijection, $A_1 \cup A_2$ is countable. □

Solution. We claim that the set of irrational real numbers is uncountable.

The real numbers are the union of rational numbers and irrational numbers. We know that $\mathbb{R}$ is uncountable. Suppose for contradiction that the set of irrational numbers is countable. Since the rationals are countable, $\mathbb{R}$ is the union of two countable sets, which by the above lemma is countable, a contradiction. Hence, the irrational numbers are uncountable. □