Analysis HW #6

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Problem (Rudin 2.16)**.** Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in $\mathbb{Q}$?

Solution. First, let's show that E is bounded in $\mathbb{Q}$. Consider the upper bound $u = 20$. Take some $e \in E$. Suppose for contradiction that $e > u$. Then $e > 20$. Then $e^2 > 400 > 3$, so $e^2 \not< 3$, so $e \notin E$, a contradiction. Now, consider the lower bound $l = -1000$. Suppose for contradiction that $\exists e \in E$ such that $e < l$. But then $e < -1000$. Then $e^2 > 1000000$. But then $e^2 > 3$, so $e^2 \not< 3$, a contradiction. Thus, we know that -1000 is a lower bound for E, and 20 is an upper bound for E. Hence, E is bounded in $\mathbb{Q}$.

Now, let's show that E is closed. Suffices to show $E^{\mathsf{c}} \in \mathbb{Q}$ is open. We know that $\mathbb{Q} \subseteq \mathbb{R}$. Let $A = (-\infty, -\sqrt{3}) \cup (-\sqrt{2}, \sqrt{2}) \cup (\sqrt{3}, \infty) \in \mathbb{R}$, an open set[1]. Then $E^{\mathsf{c}} = A \cap \mathbb{Q}$. By **Theorem 2.30** in Rudin, E^{c} is open. Hence E is closed.

Now, let's show that E is open. Take any point $p \in E$. Then $2 < p^2 < 3$. By **Problem 6** on Homework 3, we can find a $\epsilon > 0$ such that $(p+\epsilon)^2 < 3$. Also, we can find a $\delta > 0$ such that $(2+\delta)^2 < p$. Let $\gamma = \min(\delta, \epsilon)$. Then $N_\gamma(p) \subseteq E$, so p is an interior point. Hence E is open.

Now, let's show that E is not compact. We use the standard construction we used on the last homework—namely, the open cover where $G_n = \{p \in \mathbb{Q} \mid 2 < p^2 < 3 - \frac{1}{n}\}$. $\cup_{n=1}^{\infty} G_n$ covers E by the Archimedean property; however, there is no finite subcover that covers E. ☐

Problem (Rudin 2.23)**.** A collection $\{V_\alpha\}$ of open sets is said to be a *base* of X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that

[1] Since the union of open sets is open, and open intervals in $\mathbb{R}$ are open sets.

$x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X.

Solution. Let $D = \{x_1, x_2, \ldots\}$ be a countable dense subset of X. Let $V_r^n = N_r(x_n)$, where r is a positive rational (and n is a positive natural). Since the rationals are countable and the naturals are countable, there is a countable number of such V_r^n's[2]. Suffices to show that these V_r^n's form a base. So fix some $x \in X$, and some open set G such that $x \in G \subseteq X$. By the definition of an open set, we can find some $\epsilon > 0$ such that $N_\epsilon(x) \subseteq G$. Now, choose some $x_i \in D$ such that $d(x_i, x) = \gamma < \epsilon/2$, which exists by density of D. Now, since $\mathbb{Q}$ is dense in $\mathbb{R}$, choose some $q \in \mathbb{Q}$ such that $\gamma < q < \epsilon/2$. Consider the neighborhood $N_q(x_i)$, which is in the base since q is rational. Since $d(x, x_i) = \gamma < q$, $x \in V_q^i$. But any point in V_q^i is in $N(x, \epsilon)$ (via the triangle inequality), so $x \in V_q^i \subseteq N_\epsilon(x) \subseteq G$, so it is a base, as desired. □

Problem (Rudin 2.25)**.** Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n, there are finitely many neighborhoods of radius $1/n$ whose union covers K.

Solution. Since K is compact, $\forall n \in I_n$, there are finitely many neighborhoods with radius $1/n$ such that their union covers K. Take the union of all of these neighborhoods—since n ranges through every natural, and the number of neighborhoods for each n is finite, the total number of neighborhoods is countable. We claim that this forms a base. To show that it is a base, we need to show that it is countable (done) and dense. We show that in a way similar to the last problem—fix some x, and an open set G such that $x \in G \subseteq K$. By the definition of an open set, we can find some $\epsilon > 0$ such that $N_\epsilon(x) \subseteq G$. Now, similarly to the last problem, we can find a point in the base less than ϵ away from x. Hence K is dense. Thus it is a base. □

Problem (Rudin 2.26)**.** Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}, n = 1, 2, 3, \ldots$. If no finite subcollection of $\{G_n\}$ covers X, then

[2] This is true since $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality—consider the bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ via $(m, n) \mapsto 2^m(2n + 1) - 1$.

the complement F_n of $G_1 \cup \ldots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.

Solution. Rudin's hints rarely lead us astray, so let's follow the hint. By exercises 23 and 24, X has a countable base. So every open cover of X has a countable subcover $\{G_n\}$, $n \in \mathbb{N}^+$. We want to show that there is a finite subcollection of $\{G_n\}$ that covers X. Suppose not. Then the complement F_n of $G_1 \cup \ldots \cup G_n$ is nonempty for any n (since if it were empty, there would be a finite subcover). However, $\cap F_n$ is empty (since if it were nonempty, then $\{G_n\}$ wouldn't be a cover). Now let $E = \{e_1, e_2, \ldots, e_k\}$ be a set which contains a point from each F_n. Since it contains a point from each complement, $e_i \notin G_1 \cup \ldots \cup G_n$. Since E has a point all the time in infinitely many intersections, E must be infinite. Now consider some limit point x of E. Then $x \in G_i$ for some i; since each G_α is open, $\exists \epsilon > 0$ s.t. $N_\epsilon(x) \subseteq G_i$. But we know that $N_\epsilon(x)$ can't contain any other e_j for $j \geq i$! Hence x is not a limit point of E, got 'em! □

Problem (Rudin 2.28)**.** Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* Use Exercise 27.

Solution. Let K be a closed set in a separable metric space. Since it is closed, it contains K' (its limit points) as well as K^p (its condensation points). K^p is perfect by **Exercise 27**. But $K = K^p \cup (K \setminus K^p)$, where the former is a perfect set. Thus, suffices to show that $K \setminus K^p$ is at most countable. If K is countable, then K^p is empty, so we are done. So suppose K is uncountable. But then by **Exercise 27**, at most countably many points of K are not in K^p, so $K \setminus K^p$ is not uncountable. So we are done. □

Problem (7)**.** Let $a(x)$ be a function on $\mathbb{R}$ such that

(i) $a(x) \geq 0$ for all x, and

(ii) There exists $M < \infty$ such that for all *finite* $F \subseteq \mathbb{R}$,

$$\sum_F a(x) \leq M.$$

Prove $\{x \mid a(x) > 0\}$ is countable.

Solution. Let $K = \{x \mid a(x) > 0\}$. We want to show that K is countable. Suppose not. Then K is uncountable. We construct a (countably) infinite chain of subsets $K_1 \subseteq K_2 \subseteq \ldots \subseteq K$, where $K_n := \{x \mid a(x) > 1/n\}$. We want to show that $\cup_{i=1}^{\infty} K_i = K$. If not, then $\exists v \in K$ such that $v \notin K_i\ \forall i$. But by the Archimedean property, we can find some m such that $1/n < v$, so $v \in K_m$, a contradiction. Hence, $\cup_{i=1}^{\infty} K_i = K$. But by **Theorem 2.12** in Rudin, the countable union of countable (or finite) sets is countable, and K is uncountable. Thus, $\exists j$ such that K_j is uncountable. But then we can always find an arbitrarily large number in K_j by the Archimedean property, so we can get a $\sum_F a(x) > M$, a contradiction. ☐

Problem (3)**.** Prove that $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ have the same cardinality.

Solution. We construct a bijection $f : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ by $f(A, B) \to g(A) \cup h(B)$, where $g(A) := \{2a \mid a \in A\}$ and $h(B) := \{2b + 1 \mid b \in B\}$. We now show that f is a bijection.

First, let's show that f is injective. That is, if $f(A, B) \neq f(C, D)$, then $(A, B) \neq (C, D)$. Choose some $(A, B), (C, D)$ such that $f(A, B) \neq f(C, D)$. Suppose for contradiction that $(A, B) = (C, D)$. Then $A = C$ and $B = D$. Since g and h are injections, $g(A) = g(C)$ and $h(B) = h(D)$. Hence $g(A) \cup h(B) = g(C) \cup h(D)$, so $f(A, B) = f(C, D)$, a contradiction.

Now, let's show that f surjects onto $\mathcal{P}(\mathbb{N})$. That is, for any $X \in \mathcal{P}(\mathbb{N})$, we would like to find a pair $(A, B) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$ such that $f(A, B) = X$. Observe that any $X \subseteq \mathbb{N}$; hence, we can write $X = E \cup O$ where E consists of even naturals, and O consists of odd naturals[3]. Since E consists of even naturals, every $e \in E$ is in the form $2a$, $a \in \mathbb{N}$; similarly, every $o \in O$ is in the form $2b + 1$, $b \in \mathbb{N}$. Thus, let $A = \{e/2 \mid e \in E\}$, and $B = \{(o - 1)/2 \mid o \in O\}$. It follows from the form of the elements that $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$. Hence, $(A, B) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{N})$, and $g(A) = E$ and $h(B) = O$, so $g(A) \cup h(B) = E \cup O = X$, as desired. ☐

[3] We can do this since every natural number is either even or odd.