The Neeman Lectures on Analysis

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1 Lecture 15

Recall some of the things we proved in the last lecture:

Theorem 1

In $\mathbb{R}$, closed and bounded intervals are compact.

This is the proof where we kept splitting the interval in half, and choosing the problematic half. We then took a point in the union of these halves, so we isolate the problematic interval to a point, which trivially has a finite subcover for any open cover. This method of proof will be important later on.

We also showed that

Proposition 2

Compact sets are closed.

Note that this is true in any metric space, not just Euclidean space! Finally, we showed the following proposition:

Proposition 3

If K is compact, and $E \subseteq K$ is closed, then E is compact.

Ok, so now let's see what we can do with all of this. The first thing we get is the following corollary:

Corollary 4

In $\mathbb{R}$, every closed bounded[a] set E is compact.

[a](Note when we say "bounded in $\mathbb{R}$", we usually mean both above and below.)

Proof. Since E is bounded, we have some $a, b \in R$ such that $E \subseteq [a, b]$. We know by 1 that $[a, b]$ is compact, therefore closed; hence, by 3, E is compact. □

Now it would be nice to prove some properties of compactness—why is it such an important property, anyway? Well, the following set of theorems demonstrates some very nice properties that all compact sets share.

Theorem 5

If a set K is compact, then every infinite subset A of K has a limit point in K.

Proof. Suppose for contradiction that no $x \in K$ is a limit point of A. Then we have some $\delta_x > 0$ such that $N_{\delta_x}(x) \cap (A \setminus \{x\}) = \varnothing$; i.e., $N_{\delta_x}(x) \cap A \subseteq \{x\}$.

Now, let $G_x = N_{\delta_x}(x)$.

Note that G_x, $x \in K$ describes an open cover of K.

By compactness, we can find some $x_1, \ldots, x_n$ such that $G_{x_1} \cup \ldots \cup G_{x_n} \supseteq K$. Then since $A \cap G_{x_i} \subseteq \{x_i\}$, we get that

$$A = A \cap K \subseteq \cup_{i=1}^{n} (A \cap G_{x_i}) \subseteq \cup_{i=1}^{n} \{x_i\} = \{x_1, \ldots, x_n\}. \tag{1}$$

Thus, A is finite, which gives a contradiction. got 'em! □

Theorem 6

The following are equivalent in $\mathbb{R}$, for some set $E \subseteq R$:

(a) E is closed and bounded.
(b) E is compact.
(c) Every infinite subset of E has a limit point.

Proof. We showed by 1 that ① $\implies$ ②, and by 5 that ② $\implies$ ③. Thus, it suffices to show that ③ $\implies$ ①; that us, that if every infinite subset of E has a limit point, then E is closed and bounded.

Fix $E \subseteq \mathbb{R}$. Suppose every infinite subset of E has a limit point in E.

Claim 6.1. E is bounded above.

Proof. Suppose not. Then for each $n \in \mathbb{N}$, we can find an $x_n \in E$ such that $x_n > n$, since n is not an upper bound. Check (*Exercise!*) that $A = \{x_n \mid n \in \mathbb{N}\}$ has no limit points. □

Claim 6.2. E is bounded below.

Proof. The proof is similar to the one above. □

Claim 6.3. E is closed.

Note that once we prove this claim, we are done.

Proof. Let u be a limit point of E. We will construct an infinite subset $A \subseteq E$ such that u is a limit point of A, and the only one. Then by assumption, $u \in E$, so E will be closed.

To construct A:

for each n, we know that $N_{\frac{1}{n}}(u) \cap (E \setminus \{u\}) \neq \varnothing$, since u is a limit point of E.

Now pick some $x_n \in N_{\frac{1}{n}}(u) \cap (E \setminus \{u\})$. Set $A = \{x_n \mid n \in \mathbb{N}\}$.

We need to show the following two things:

- u is a limit point of A (which implies the infinitude of A) (*Exercise!*)
- u is the only limit point of A (*Exercise!*)

□

□

Corollary 7

In $\mathbb{R}$, every bounded infinite set has a limit point.

Note 8

It's not hard to prove the same result in $\mathbb{R}^k$. (*Optional exercise*)